Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5113: Introduction to Differential Geometry

(Solutions)

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Shao, B. Noohi

Question 1 [17 marks]. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + y^2 = 16\},$$

and consider the following parametrisation of C:

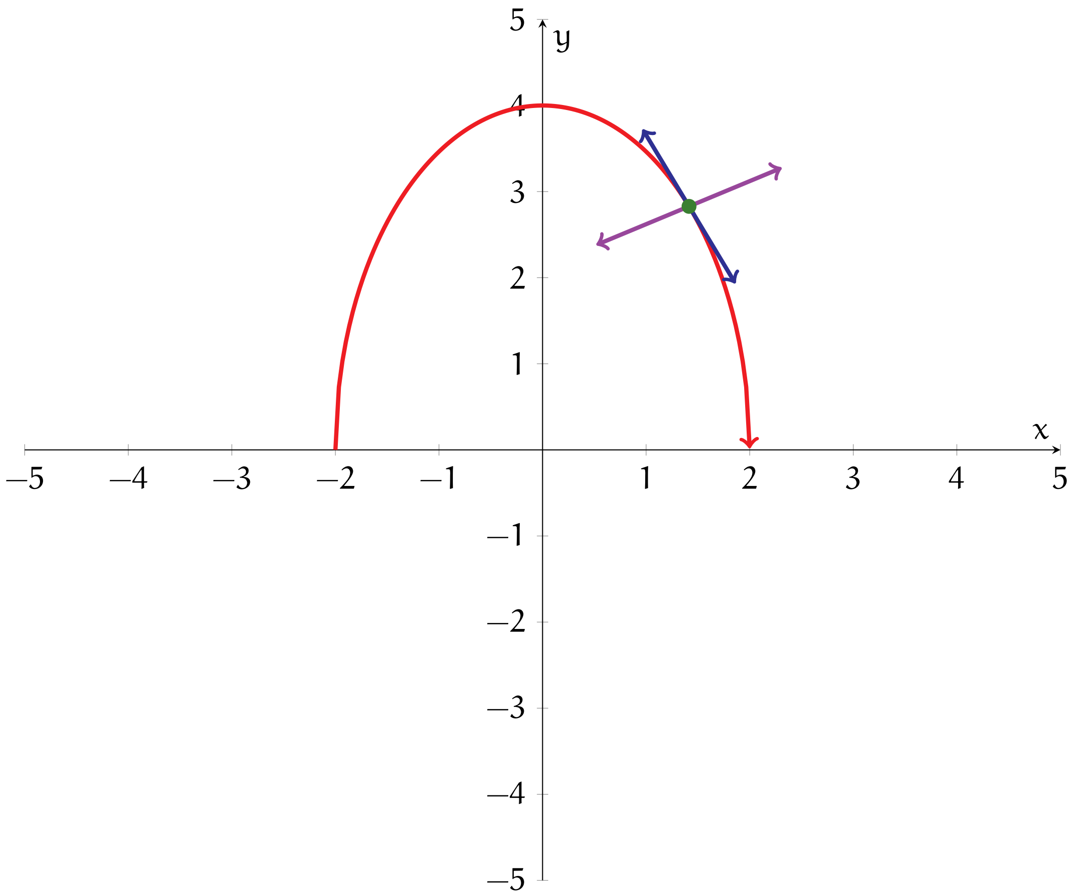
$$\gamma : (-2, 2) \to C, \qquad \gamma(t) = \left(t,\ \sqrt{16 - 4t^2}\right).$$

(a) Sketch the image of γ. [5]

(b) Find the unit tangents and the unit normals to C at the point $\left(\sqrt{2},\ 2\sqrt{2}\right)$. **Draw and label these on your sketch from part (a).** [8]

(c) Can the parametrisation γ be used to directly compute the tangent line to C at $(0, -4)$? Briefly explain why or why not. [4]

(a) [Seen similar] The image of γ is drawn in red. [5 marks] (Here, one needs not be exact—getting the general shape of γ along with a few key values of γ would suffice.)



(b) [Seen similar] First, a direct computation yields

$$\gamma'(t) = \left(1, -\frac{4t}{\sqrt{16 - 4t^2}}\right). \quad \text{[1 mark]}$$

Furthermore, observe that

$$\gamma(\sqrt{2}) = (\sqrt{2}, 2\sqrt{2}), \qquad \gamma'(\sqrt{2}) = (1, -2), \qquad |\gamma'(\sqrt{2})| = \sqrt{5}. \quad \text{[2 marks]}$$

As a result, the unit tangents are given by

$$\mathbf{t}^{\pm} = \pm\frac{1}{|\gamma'(\sqrt{2})|} \cdot \gamma'(\sqrt{2})_{\gamma(\sqrt{2})} = \pm\frac{1}{\sqrt{5}}(1, -2)_{(\sqrt{2}, 2\sqrt{2})}. \quad \text{[2 marks]}$$

The unit normals are then obtained by rotating the unit tangents:

$$\mathbf{n}^{\pm} = \pm\frac{1}{\sqrt{5}}(2, 1)_{(\sqrt{2}, 2\sqrt{2})}. \quad \text{[1 mark]}$$

Alternatively, one can note that C is a level set of the function $f(x, y) = 4x^2 + y^2$, defined on all of $\mathbb{R}^2$. Taking a gradient of f, at each $(x, y) \in \mathbb{R}^2$, yields

$$\nabla f(x, y) = (8x, 2y)_{(x,y)}. \quad \text{[2 marks]}$$

Moreover, note that

$$\nabla f(\sqrt{2}, 2\sqrt{2}) = 4\sqrt{2}\,(2, 1)_{(\sqrt{2}, 2\sqrt{2})}, \qquad |\nabla f(\sqrt{2}, 2\sqrt{2})| = 4\sqrt{2} \cdot \sqrt{5}. \quad \text{[2 marks]}$$

As a result, the unit normals are given by

$$\mathbf{n}^{\pm} = \pm\frac{1}{|\nabla f(\sqrt{2}, 2\sqrt{2})|}\nabla f(\sqrt{2}, 2\sqrt{2}) = \pm\frac{1}{\sqrt{5}}(2, 1)_{(\sqrt{2}, 2\sqrt{2})}, \quad \text{[1 mark]}$$

and the unit tangents can then obtained by rotating the above

$$\mathbf{t}^{\pm} = \pm\frac{1}{|\gamma'(\sqrt{2})|} \cdot \gamma'(\sqrt{2})_{\gamma(\sqrt{2})} = \pm\frac{1}{\sqrt{5}}(1, -2)_{(\sqrt{2}, 2\sqrt{2})}. \quad \text{[1 mark]}$$

Finally, the unit tangents and unit normals are drawn in blue and purple, respectively, on the sketch in part (a). [2 marks]

(c) [Unseen] No. [1 mark] The point $(0, -4)$ does not lie in the image of γ, thus the tangent line at $(0, -4)$ cannot be extracted from γ. [3 marks]

Question 2 [23 marks]. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < 2,\ 4y^2 + 4(z+1)^2 = x^2\},$$

and consider the following parametrisation of S:

$$\sigma : (0, 1) \times \mathbb{R} \to S, \qquad \sigma(u, v) = (2u,\ u\cos v,\ u\sin v - 1).$$

(a) Sketch the image of σ. Moreover, on your sketch, indicate (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. [**8**]

(b) Find the tangent plane to S at the point $\left(1, \frac{1}{2}, -1\right)$. [**6**]

(c) Compute the surface integral

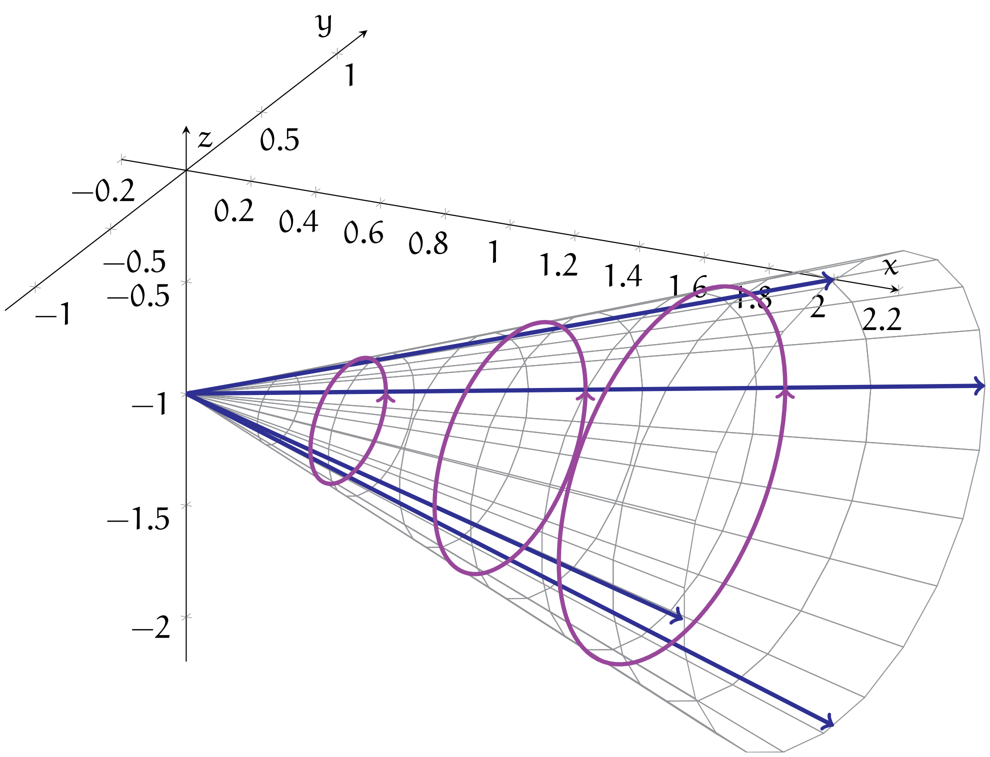
$$\iint_S \mathbf{H} \cdot \mathbf{dA},$$

where S has the outward-facing orientation, and where $\mathbf{H}$ is the vector field

$$\mathbf{H}(x, y, z) = (x^2,\ -z-1,\ y)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$ [**9**]

(Hint: The image of σ is all of S.)

(a) [Seen similar] The image of σ is drawn below [4 marks]; examples of level paths are drawn in (i) blue [2 marks] and (ii) purple [2 points], respectively. (Here, one needs not be exact—getting the general shape of σ along with a few key values of σ would suffice.)



(b) [Seen similar] The partial derivatives of σ satisfy

$$\partial_1\sigma(u,v) = (2,\ \cos v,\ \sin v), \qquad \partial_2\sigma(u,v) = (0,\ -u\sin v,\ u\cos v). \quad \text{[2 mark]}$$

Moreover, notice that

$$\sigma\left(\frac{1}{2}, 0\right) = \left(1,\ \frac{1}{2},\ -1\right),$$

and that

$$\partial_1\sigma\left(\frac{1}{2}, 0\right) = (2,\ 1,\ 0), \qquad \partial_2\sigma\left(\frac{1}{2}, 0\right) = \left(0,\ 0,\ \frac{1}{2}\right) \quad \text{[3 marks]}$$

Thus, by definition, the tangent plane is given by

$$\begin{aligned} T_{(1,-\frac{1}{2},-1)}S &= T_\sigma\left(\frac{1}{2},\ 0\right) \\ &= \left\{ a\cdot(2,\ 1,\ 0)_{(1,\frac{1}{2},-1)} + b\cdot\left(0, 0, \frac{1}{2}\right)_{(1,\frac{1}{2},-1)} \middle| a, b \in \mathbb{R} \right\}. \quad \text{[1 mark]} \end{aligned}$$

(c) [Seen similar] First, we further restrict the domain of σ:

$$\rho : (0,1)\times(0,2\pi) \to S, \qquad \rho(u,v) = (2u,\ u\cos v,\ u\sin v - 1).$$

Note ρ is injective, and its image is all of S except for a single curve ($u = 0$). [2 marks]

Next, direct computations show that

$$\partial_1\rho(u,v) = (2,\ \cos v,\ \sin v), \qquad \partial_2\rho(u,v) = (0,\ -u\sin v,\ u\cos v).$$

Furthermore, we can calculate

$$\begin{aligned} \mathbf{H}(\rho(u,v)) &= (4u^2,\ -u\sin v,\ u\cos v)_{\rho(u,v)}, \\ [\partial_1\rho(u,v)\times\partial_2\rho(u,v)]_{\rho(u,v)} &= (u,\ -2u\cos v, -2u\sin v)_{\rho(u,v)}. \quad \text{[3 marks]} \end{aligned}$$

By inspection, we see that $[\partial_1\rho(u,v)\times\partial_2\rho(u,v)]_{\rho(u,v)}$ points inward from S, thus ρ generates the orientation opposite to our given outward orientation. [2 marks].

Finally, combining the above, we compute that

$$\begin{aligned} \iint_S \mathbf{H}\cdot d\mathbf{A} &= -\iint_{(0,1)\times(0,2\pi)} [(4u^2,\ -u\sin v,\ u\cos v)\cdot(u,\ -2u\cos v,\ -2u\sin v)]\,du dv \\ &= -\int_0^{2\pi} dv \int_0^1 4u^3\,du \\ &= -2\pi. \quad \text{[2 marks]} \end{aligned}$$

Question 3 [19 marks].

(a) Find the minimum and maximum values of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x - y,$$

subject to the constraint

$$x^4 + y^4 = 32.$$

Also, at which points are these minimum and maximum values achieved? **[15]**

(b) Dr Mistake wishes to find the maximum and minimum values of $a(x, y) = x$, subject to the constraint $b(x, y) = x^2 - y^2 = 1$. For this, Dr Mistake applies the method of Lagrange multipliers to solve the system

$$\nabla a(x, y) = \lambda \, \nabla b(x, y), \qquad b(x, y) = 1,$$

and correctly obtains the solutions $(x, y, \lambda) = \pm\left(1, 0, \frac{1}{2}\right)$ to this system. However, Dr Mistake then incorrectly concludes that the maximum and minimum values of $a(x, y)$ are ± 1 and are achieved at $(x, y) = \pm(1, 0)$. What did Dr Mistake do wrong? Explain Dr Mistake's mistake. **[4]**

(a) [Seen similar] First, notice that the constraint curve $x^4 + y^4 = 1$ is both closed and bounded, therefore the maximum and minimum values in this constrained optimisation problem are guaranteed to exist. [3 marks]

Let g denote the function corresponding to the constraint:

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^4 + y^4.$$

First, we compute the gradients of f and g:

$$\nabla f(x, y) = (1, \, -1)_{(x,y)}, \qquad \nabla g(x, y) = (4x^3, \, 4y^3)_{(x,y)}.$$

The method of Lagrange multipliers then indicates that we must solve the system,

$$1 = \lambda \cdot 4x^3, \qquad -1 = \lambda \cdot 4y^3, \qquad x^4 + y^4 = 32.$$ [3 marks]

From the first or second equations of the system, we obtain that $\lambda \neq 0$, since otherwise we obtain a contradiction $1 = 0$ or $-1 = 0$. [2 marks]

With this in mind, we can divide by 4λ in the first two equations to obtain

$$x^3 = \frac{1}{4\lambda} = -y^3, \qquad x = -y.$$

Combining this with the third equation yields

$$32 = x^4 + y^4 = 2y^4, \qquad y = \pm 2,$$

while the above then implies

$$x = -y = \mp 2, \qquad \lambda = \frac{1}{4x^3} = \mp \frac{1}{32}.$$

From all of this, we conclude that there are two solutions to our system:

$$(x, y, \lambda) = \left(-2, +2, -\frac{1}{32}\right), \left(+2, -2, +\frac{1}{32}\right). \quad \text{[4 marks]}$$

We can now evaluate f at each of the above points:

$$f(-2, +2) = -4, \qquad f(+2, -2) = +4. \quad \text{[1 mark]}$$

Since the extrema of f are guaranteed to exist, we can hence conclude that:

- The maximum value is $+4$, and this is achieved at $(x, y) = (+2, -2)$. [1 mark]
- The minimum value is -4, and this is achieved at $(x, y) = (-2, +2)$. [1 mark]

(b) [Seen similar] The constraint curve $b(x, y) = 1$ (a hyperbola) is not bounded, thus the maximum and minimum values need not exist for this constrained optimisation problem. [4 marks] (In fact, $a(x, y)$ achieves neither a maximum value nor a minimum value when subject to the constraint $b(x, y) = 1$.)

Question 4 [20 marks].

(a) Show that the following set is a curve:

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 3xy + 2y^2 = 1\}.$$

Briefly explain your reasoning. [5]

(b) Compute the arc length of the curve

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1,\ x = 2e^z,\ y = e^{2z}\}.$$ [6]

(c) Let **F** be a vector field on $\mathbb{R}^3$. Dr Mistake wishes to compute the curl of the divergence of **F** (i.e. $\nabla \times (\nabla \cdot \mathbf{F})$). Why is Dr Mistake not allowed to do this? [4]

(d) Let C denote the unit circle about the origin,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

with the **anticlockwise** orientation, and let **W** be the vector field on $\mathbb{R}^2$ given by

$$\mathbf{W}(x, y) = (ax + by, cx + dy)_{(x,y)},$$

where $a, b, c, d \in \mathbb{R}$ are fixed constants. Apply **Green's theorem** to compute

$$\int_C \mathbf{W} \cdot ds,$$

State your answer in terms of the constants a, b, c, d. [5]

(a) [Seen similar] First, note that M can be expressed as the level set

$$M = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 1\}, \qquad f(x, y) = x^2 - 3xy + 2y^2.$$

The gradient of f satisfies

$$\nabla f(x, y) = (2x - 3y,\ 4y - 3x)_{(x,y)}, \quad \text{[2 marks]}$$

which, by a bit of linear algebra, vanishes only when $(x, y) = (0, 0)$. [2 marks] Since $(0, 0) \notin M$, it follows (from the level set theorem) that M is indeed a curve. [1 mark]

(b) [Seen similar] First, note that the function

$$\lambda : (0, 1) \to X, \qquad \lambda(t) = (2e^t,\ e^{2t},\ t)$$

is an injective parametrisation of X, whose image is all of X. [2 marks] Moreover,

$$\lambda'(t) = (2e^t,\ 2e^{2t},\ 1), \qquad |\lambda'(t)| = \sqrt{4e^{4t} + 4e^{2t} + 1} = 1 + 2e^{2t}. \quad \text{[2 marks]}$$

Finally, we use λ to compute the desired arc length:

$$L(X) = \int_0^1 |\lambda'(t)|\,dt = \int_0^1 (1 + 2e^{2t})\,dt = (t + e^{2t})|_{t=0}^{t=1} = e^2. \quad \text{[2 marks]}$$

(c) [Unseen] The divergence of **F** is a real-valued function, but the curl is only applied to vector fields. Thus, the curl of the divergence of **F** is not defined. [4 marks]

(d) [Seen similar] Let D denote the interior of C:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Applying Green's theorem, we see that

$$\int_C \mathbf{W} \cdot d\mathbf{s} = \iint_D [\partial_x(cx + dy) - \partial_y(ax + by)]\,dx\,dy. \quad \text{[3 marks]}$$

Recalling that the area of D is $\pi \cdot 1^2 = \pi$, we then conclude

$$\int_C \mathbf{W} \cdot d\mathbf{s} = (c - b) \iint_D dx\,dy = (c - b)\pi. \quad \text{[2 marks]}$$

Question 5 [21 marks].

(a) Is the following parametric surface regular:

$$\rho : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \rho(u, v) = (u \cos v,\ u \sin v,\ u^4)?$$

Justify your answer. **[5]**

(b) Find the unit normals at the point $\left(\sqrt{2}, -\sqrt{2}, \sqrt{2}\right)$ to the hyperboloid

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid 4z^2 = x^2 + y^2 + 4\}.$$ **[6]**

(c) Let Q be the quarter-sphere,

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x > 0,\ y > 0\}.$$

Describe the boundary of this surface Q, as one or more curves in $\mathbb{R}^3$. (You should express these curves as sets, or you can draw them for partial credit.) **[5]**

(d) Recall, from discussions in the lectures or lecture notes, that one can apply Green's theorem in order to express the area of an open region $D \subseteq \mathbb{R}^2$ in terms of integrals over the boundary of D.

Describe how one can similarly apply the divergence theorem to express the volume of an open region $V \subseteq \mathbb{R}^3$ in terms of an integral over the boundary of V. Here, you can suppose that the boundary of V is a single surface S. **[5]**

(a) [Seen similar] We begin by differentiating ρ:

$$\partial_1\rho(u, v) = (\cos v,\ \sin v,\ 4u^3), \qquad \partial_2\rho(u, v) = (-u \sin v,\ u \cos v,\ 0).$$ [2 marks]

In particular,

$$\partial_1\rho(u, v) \times \partial_2\rho(u, v) = (-4u^4 \cos v,\ -4u^4 \sin v,\ u),$$

which vanishes whenever $u = 0$. [2 marks] Thus, ρ is not regular. [1 mark]

(b) [Seen similar] Note that H can be written as the level set

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = 4\}, \qquad h(x, y, z) = 4z^2 - x^2 - y^2.$$

Moreover, the gradient of h satisfies

$$\nabla h(x, y, z) = (-2x, -2y, 8z)_{(x,y,z)}.$$ [2 marks]

Note that in particular,

$$\nabla h\left(\sqrt{2},\ -\sqrt{2},\ \sqrt{2}\right) = \left(-2\sqrt{2},\ 2\sqrt{2},\ 8\sqrt{2}\right)_{(\sqrt{2},-\sqrt{2},\sqrt{2})},$$

$$\left|\nabla h\left(\sqrt{2},\ -\sqrt{2},\ \sqrt{2}\right)\right| = 12.$$ [2 marks]

As a result, the unit normals to H are given by

$$\begin{aligned} \mathbf{n}^{\pm} &= \pm \frac{1}{|\nabla h(\sqrt{2}, -\sqrt{2}, \sqrt{2})|} \nabla h(\sqrt{2}, -\sqrt{2}, \sqrt{2}) \\ &= \pm \frac{\sqrt{2}}{6} (-1, 1, 4)_{(\sqrt{2}, -\sqrt{2}, \sqrt{2})}. \quad \text{[2 marks]} \end{aligned}$$

(c) [Seen similar] The boundary of Q consists of two semi-circles:

$$\begin{aligned} C_1 &= \{(x, 0, z) \in \mathbb{R}^2 \mid x^2 + z^2 = 1,\ x > 0\}, \\ C_2 &= \{(0, y, z) \in \mathbb{R}^2 \mid y^2 + z^2 = 1,\ y > 0\}. \end{aligned}$$

[5 marks] ([3 marks for correct drawing])

(d) [Unseen] Let $\mathbf{F}$ denote the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = \frac{1}{3} (x, y, z)_{(x,y,z)}.$$

Note that

$$(\nabla \cdot \mathbf{F})(x, y, z) = 1, \qquad (x, y, z) \in \mathbb{R}^3. \quad \text{[2 marks]}$$

Here, $\mathbf{F}$ can be replaced by any vector field with divergence equal to 1.

Applying the divergence theorem, we then see that

$$\iiint_V dx\,dy\,dz = \iint_V (\nabla \cdot \mathbf{F})\, dx\,dy\,dz = \iint_S \mathbf{F} \cdot d\mathbf{A}.$$

hence the volume of V is expressed as the integral of $\mathbf{F}$ over S. [3 marks]

End of Paper.