Queen Mary University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH5113: Introduction to Differential Geometry

(Solutions)

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Shao, S. Majid

There is a compendium of definitions and formulae in the appendix, which you are free to use without comment.

Question 1 [20 marks]. Consider the curve

$$C = \{(t\cos t, t\sin t, -t) \in \mathbb{R}^3 \mid 0 < t < 4\pi\},$$

and consider the following parametrisation of C:

$$\gamma : (0, 4\pi) \to C, \qquad \gamma(t) = (t\cos t, t\sin t, -t).$$

(a) Sketch the image of γ. [**6**]

(b) Find the tangent line to C at the point $(-3\pi, 0, -3\pi)$. **Draw this on your sketch from part (a).** [**7**]

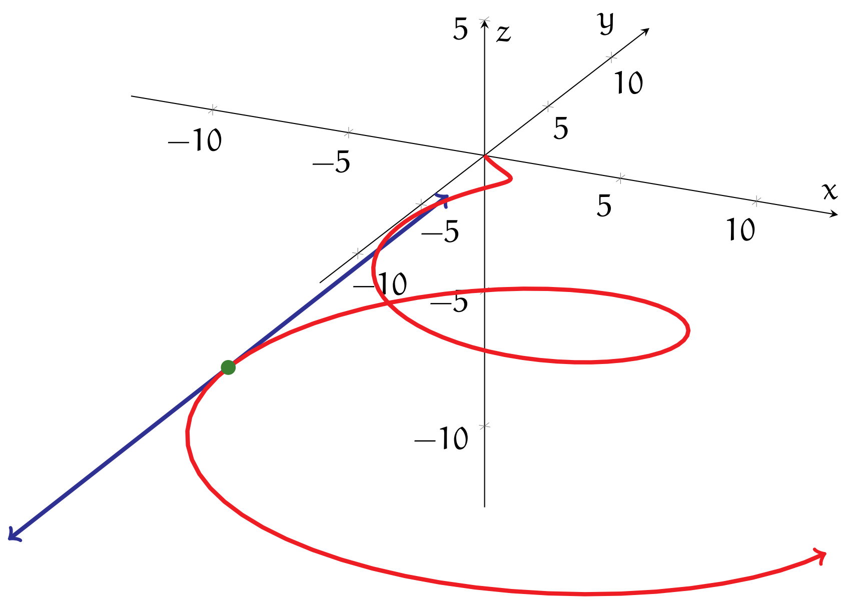
(c) Compute the curve integral

$$\int_C \mathbf{F} \cdot \mathbf{ds},$$

where C is given the **upward** orientation (in the direction of **increasing** z**-value**), and where $\mathbf{F}$ is the vector field given by

$$\mathbf{F}(x, y, z) = (y, -x, z^2)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$ [**7**]

(a) [Seen similar] γ is drawn in red. [6 points] (Here, one needs not be exact—getting the general shape of γ along with a few key values of γ would suffice.)



(b) [Seen similar] Direct computations yield

$$\gamma'(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ -1). \quad \text{[2 points]}$$

Furthermore, observe that

$$(-3\pi, 0, -3\pi) = \gamma(3\pi), \qquad \gamma'(3\pi) = (-1, -3\pi, -1). \quad \text{[2 point]}$$

As a result, we conclude that

$$\begin{aligned} T_{(-3\pi,0,-3\pi)}C &= T_\gamma(3\pi) \\ &= \{s \cdot (-1, -3\pi, -1)_{(-3\pi,0,-3\pi)} \mid s \in \mathbb{R}\}. \quad \text{[1 point]} \end{aligned}$$

$T_{(-3\pi,0,-3\pi)}C$ is drawn in blue on the sketch in part (a). [2 point]

(c) [Seen similar] First, observe γ is injective, and its image is all of C. Moreover, γ generates the downward orientation of C, opposite of what we are given. [2 points].

Next, we compute the necessary quantities:

$$\begin{aligned} \mathbf{F}(\gamma(t)) &= (t\sin t,\ -t\cos t,\ t^2)_{(t\cos t,\, t\sin t,\, -t)}, \\ \gamma'(t) &= (\cos t - t\sin t,\ \sin t + t\cos t,\ -1), \quad \text{[2 points]} \end{aligned}$$

Thus, putting all the above together, we compute that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{s} &= -\int_0^{4\pi} [(t\sin t,\ -t\cos t,\ t^2) \cdot (\cos t - t\sin t,\ \sin t + t\cos t,\ -1)]\, dt \\ &= -\int_0^{4\pi} (-2t^2)\, dt \\ &= \frac{2}{3} t^3 \Big|_{t=0}^{t=4\pi} \\ &= \frac{128\pi^3}{3}. \quad \text{[3 points]} \end{aligned}$$

Question 2 [16 marks].

(a) Is the following parametric curve regular:

$$\alpha : (0, \infty) \to \mathbb{R}^2, \qquad \alpha(t) = (e^t, \ln t)?$$

Justify your answer. **[5]**

(b) Is the following set a curve:

$$L = \{(x, y) \in \mathbb{R}^2 \mid (x - y)^2 = 0\}?$$

Briefly justify your answer. **[5]**

(c) Find the unit normals to the curve

$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^3 = 1\}$$

at the point $(1, 0)$. **[6]**

(a) [Seen similar] Differentiating α yields

$$\alpha'(t) = (e^t, t^{-1}).\text{[2 points]}$$

Neither e^t nor t^{-1} vanishes for any $t > 0$, thus α is indeed regular. [3 points]

(b) [Unseen] Observe that L is simply the line

$$L = \{(x, y) \in \mathbb{R}^2 \mid x = y\},$$

which is indeed a curve. [5 points] (To prove this, one can note that the gradient of $y - x$ is nowhere vanishing, however this is not necessary for full marks.)

(c) [Seen similar] Note that P is the level set

$$P = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 1\}, \qquad g(x, y) = x^2 + y^3. \quad \text{[2 point]}$$

Moreover, direct computations show that

$$\nabla g(x, y) = (2x,\ 3y^2)_{(x,y)}, \qquad \nabla g(1, 0) = (2, 0)_{(1,0)}, \qquad |\nabla g(1, 0)| = 2. \quad \text{[2 points]}$$

As a result, the unit normals to P at $(1, 0)$ are

$$\pm \mathbf{n}_{(1,0)} = \pm \frac{\nabla g(1, 0)}{|\nabla g(1, 0)|} = \pm (1, 0)_{(1,0)}. \quad \text{[2 points]}$$

Question 3 [21 marks]. Consider the surface

$$S = \{(u, u, u^2 v) \in \mathbb{R}^3 \mid 0 < u < 1,\ 0 < v < 1\},$$

and consider the following parametrisation of S:

$$\sigma : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \sigma(u, v) = (u,\ u,\ u^2 v).$$

(a) Sketch the image of σ. On your sketch, draw (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. [8]

(b) Find the tangent plane to S at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{8})$. [6]

(c) Compute the surface integral

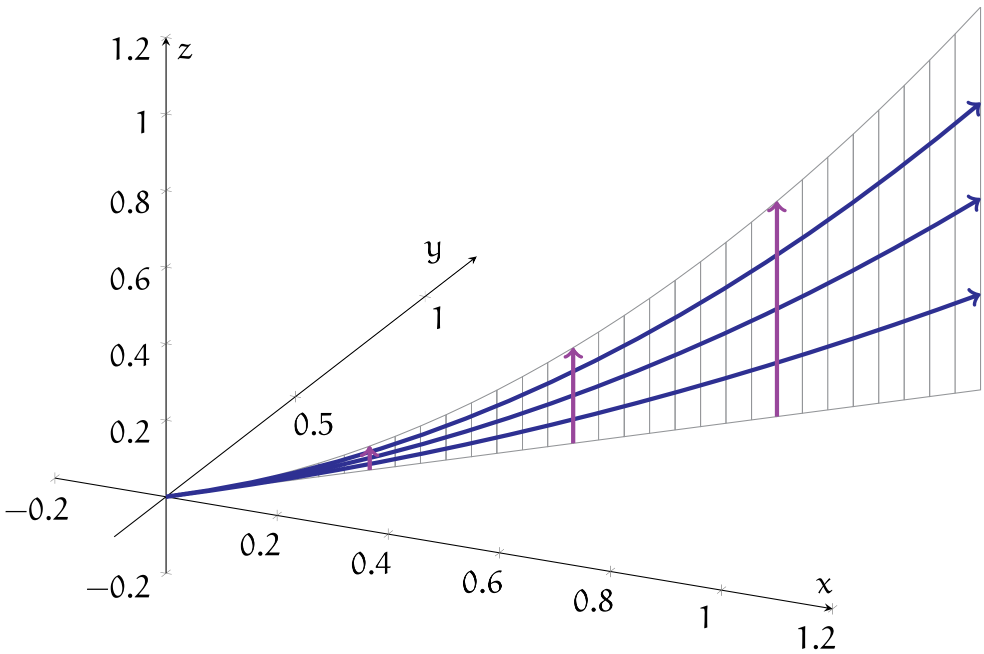
$$\iint_S H \, dA,$$

where H is the function

$$H : \mathbb{R}^3 \to \mathbb{R}, \qquad H(x, y, z) = xyz.$$ [7]

(a) [Seen similar] The image of σ is drawn in the diagram below [4 points]; examples of level paths are drawn in (i) blue and (ii) purple, respectively [4 points].

(Here, one needs not be exact—getting the general shape of σ along with a few key values of σ would suffice.)



(b) [Seen similar] First, notice that

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{8}\right) = \sigma\left(\frac{1}{2},\frac{1}{2}\right). \quad \text{[2 point]}$$

The partial derivatives of σ satisfy

$$\begin{aligned} \partial_1\sigma(u,v) &= (1,1,2uv), & \partial_2\sigma(u,v) &= (0,0,u^2), \\ \partial_1\sigma\left(\frac{1}{2},\frac{1}{2}\right) &= \left(1,1,\frac{1}{2}\right), & \partial_2\sigma\left(\frac{1}{2},\frac{1}{2}\right) &= \left(0,0,\frac{1}{4}\right). \quad \text{[3 points]} \end{aligned}$$

Thus, by definition, the tangent plane is given by

$$\begin{aligned} T_{(0,-3,0)}S &= T_\sigma\left(\frac{1}{2},\frac{1}{2}\right) \\ &= \left\{ a\cdot\left(1,1,\frac{1}{2}\right)_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{8}\right)} + b\cdot\left(0,0,\frac{1}{4}\right)_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{8}\right)} \,\middle|\, a,b\in\mathbb{R} \right\} \\ &= \left\{ a\cdot(1,1,0)_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{8}\right)} + b\cdot(0,0,1)_{\left(\frac{1}{2},\frac{1}{2},\frac{1}{8}\right)} \,\middle|\, a,b\in\mathbb{R} \right\}. \quad \text{[1 point]} \end{aligned}$$

(c) [Seen similar] First, observe σ is injective, and its image is all of S. Thus, we can use σ to compute our area integral. [2 points]

Next, we note the following intermediate computations:

$$\begin{aligned} H(\sigma(u,v)) &= u^4 v, \\ |\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)| &= |(u^2,-u^2,0)| = \sqrt{2}\,u^2. \quad \text{[2 points]} \end{aligned}$$

Combining the above, we compute that

$$\begin{aligned} \iint_S H\,dA &= \sqrt{2}\iint_{(0,1)\times(0,1)} u^4 v\cdot u^2\,du\,dv \\ &= \sqrt{2}\int_0^1 v\,dv\int_0^1 u^6\,du \\ &= \frac{\sqrt{2}}{14}. \quad \text{[3 points]} \end{aligned}$$

Question 4 [16 marks].

(a) Show that the following parametric surface is regular:

$$\rho : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \rho(u, v) = (u + v, u - v, u^3).$$ [5]

(b) Show that the following is a surface:

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 1\}.$$ [6]

Briefly explain your reasoning.

(c) Find a parametrisation of the surface,

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid z = e^x + e^{2y}\},$$

such that its image is all of Z. **Be sure to specify its domain.** [5]

(a) [Seen similar] We begin by differentiating ρ:

$$\partial_1\rho(u, v) = (1, 1, 3u^2), \qquad \partial_2\rho(u, v) = (1, -1, 0).$$ [2 points]

In particular,

$$\begin{aligned} |\partial_1\rho(u, v) \times \partial_2\rho(u, v)| &= |(3u^2,\ 3u^2,\ -2)| \\ &= \sqrt{4 + 18u^4} \geq \sqrt{4}, \end{aligned}$$ [2 points]

which never zero for any $(u, v) \in \mathbb{R}^2$. Thus, ρ is indeed regular. [1 point]

(b) [Seen similar] Note that Q can be written as the level set

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = 1\}, \qquad h(x, y, z) = x - y.$$ [1 point]

In particular, the gradient of h,

$$\nabla h(x, y, z) = (1,\ -1,\ 0)_{(x,y,z)},$$

which never vanishes. [4 points]

Thus, by the level set theorem, Q is a surface. [1 point]

(c) [Unseen] A simple solution is to let the parameters refer to the x- and y-coordinates:

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (u,\ v,\ e^u + e^{2v}).$$

[3 points for a correct parametrisation] [2 point for correct domain]

Question 5 [17 marks].

(a) Sketch the following curve:

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^4 = 1\}.$$

(Just the approximate shape of C is fine; you do not need to be very precise.) [**5**]

(b) Using the method of Lagrange multipliers, find the maximum and minimum values of the function

$$a : \mathbb{R}^2 \to \mathbb{R}, \qquad a(x, y) = x + y^2,$$

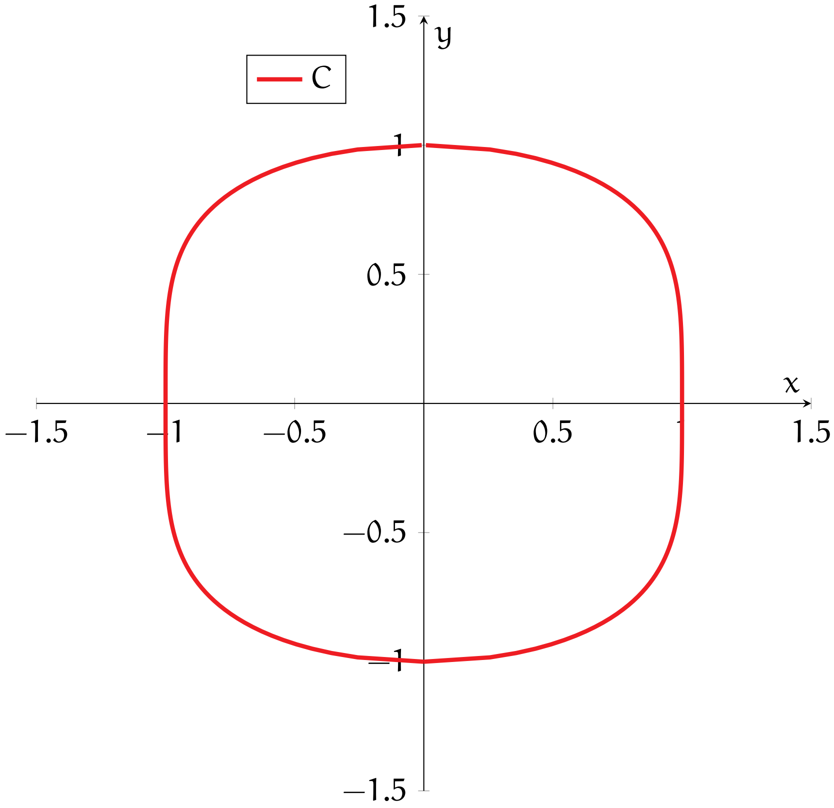
subject to the constraint

$$x^2 + y^4 = 1.$$ [**12**]

(a) [Unseen] Note that C is characterised by the relations

$$x = \pm\sqrt{1 - y^4}$$

From the above information, we can plot C as follows:
[2 points if figure is bounded and closed] [3 points for nearly correct shape]



(b) [Seen similar] Let h denote the function corresponding to the constraint:

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad h(x, y) = x^2 + y^4.$$

First, we compute the gradients of a and h:

$$\nabla a(x, y) = (1, 2y)_{(x,y)}, \qquad \nabla h(x, y) = (2x, 4y^3)_{(x,y)}.$$

The method of Lagrange multipliers indicates that we must solve the system,

$$1 = \lambda \cdot 2x, \qquad 2y = \lambda \cdot 4y^3, \qquad x^2 + y^4 = 1. \quad [3 \text{ points}]$$

From the first equation, we see that

$$x \neq 0, \qquad \lambda = \frac{1}{2x} \neq 0.$$

For the second equation, we split into two cases:

- First, if $y = 0$ (so the second equation holds), then the constraint gives us

  $$x^2 = 1, \qquad x = \pm 1.$$

  This gives us two solutions to our system:

  $$(x, y, \lambda) = \pm \left(1, 0, \frac{1}{2} \right).$$

- Next, if $y \neq 0$, then the second and third equations give

  $$y^2 = \frac{1}{2\lambda} = x, \qquad 1 = x^2 + y^4 = 2x^2.$$

  Solving the above yields

  $$x = y^2 = \pm \frac{1}{\sqrt{2}}.$$

  Since y^2 is non-negative, this yields the following solutions to our system:

  $$(x, y, \lambda) = \left(\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt{2}} \right).$$

The above four values of (x, y) are all the solutions to our system. [5 points]

Since the constraint curve $x^2 + y^4 = 1$ is closed and bounded (this can be seen from the sketch in part (a)), the maximum and minimum values of a can be found by checking its values at the points corresponding to the solutions of our system: [1 point]

$$a(+1, 0) = 1, \qquad a(-1, 0) = -1,$$
$$a\left(\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt[4]{2}} \right) = \sqrt{2}, \qquad a\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt[4]{2}} \right) = \sqrt{2}.$$

Thus, the maximum and minimum values of a are

$$a_{\max} = \sqrt{2}, \qquad a_{\min} = -1. \quad [3 \text{ points}]$$

Question 6 [10 marks].

(a) Is the following an open subset of $\mathbb{R}^2$:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ y \geq 0\}?$$

Briefly justify your answer. **[5]**

(b) Compute the divergence of the following vector field on $\mathbb{R}^3$:

$$\mathbf{G}(x, y, z) = (xe^x + yz,\ y^4,\ \cos z + \sin z)_{(x,y,z)}.$$ **[2]**

(c) Bob wants to integrate a vector field $\mathbf{F}$ (on $\mathbb{R}^3$) over a cylinder of height 1,

$$S = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ 0 < z < 1\},$$

which is also assigned the **outward-facing** orientation. Applying the **divergence theorem**, Bob obtains

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iiint_V (\nabla \cdot \mathbf{F})\, dx\, dy\, dz,$$

where V is the interior of the cylinder,

$$V = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ 0 < z < 1\}.$$

As a tutor for **MTH5113**, you see that Bob's answer is incorrect, and you decide that Bob must lose some marks for this. Where did Bob make a mistake? **[3]**

(a) [Seen similar] A is not an open subset of $\mathbb{R}^2$. [1 point]
For instance, note that the point $(0, 0)$ lies in A. On the other hand, any point $(0, y)$, where $y < 0$, is not in A. Thus, A violates the definition of open subsets. [4 points]

In other words, if you stand at $(0, 0) \in A$, and you take an arbitrarily small step downwards, then you end up outside of A.

(b) [Seen similar] This is a direct computation:

$$\begin{aligned}(\nabla \cdot \mathbf{G})(x, y, z) &= \partial_x(xe^x + yz) + \partial_y(y^4) + \partial_z(\cos z + \sin z)\\ &= e^x + xe^x + 4y^3 - \sin z + \cos z. \quad \text{[2 points]}\end{aligned}$$

(c) [Unseen] Bob applied the divergence theorem incorrectly—the boundary of V is not only the cylinder S, but also the two disks comprising the "top" and the "bottom" of V. Thus, the integrals over these disks must also be included. [3 points]

End of Paper.