Queen Mary
University of London

Main Examination period 2019

MTH5113: Introduction to Differential Geometry

Duration: 2 hours

(Solutions)

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: A. Shao, S. Majid

There is a compendium of definitions and formulae in the appendix, which you are free to use without comment.

Question 1. [22 marks] Let C be the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x+3)^2 + 4(y-2)^2 = 16\}$$

and consider the following parametrisation of C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (-3 + 4\cos t, 2 + 2\sin t).$$

(a) Find the curvature of C at the point $(-3, 0)$. [6]

(b) Sketch the image of γ, and indicate the point $(-3, 0)$ on your sketch. [5]

(c) At which points of C does its curvature achieve its **maximum** value? Justify your answer(s) computationally. [4]

(d) Compute the curve integral

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

where C has the **clockwise** orientation, and where $\mathbf{F}$ is the vector field given by

$$\mathbf{F}(x, y) = (-y, x)_{(x,y)}, \qquad (x, y) \in \mathbb{R}^2.$$ [7]

(a) [Seen similar] Direct computations for γ yield

$$\gamma'(t) = (-4\sin t, 2\cos t), \qquad \gamma''(t) = (-4\cos t, -2\sin t), \qquad [2 \text{ point}]$$

so the curvature of γ satisfies

$$\begin{aligned}\kappa_\gamma(t) &= \frac{|(-4\sin t)(-2\sin t) - (2\cos t)(-4\cos t)|}{|(-4\sin t, 2\cos t)|^3} \\ &= \frac{8}{(4\cos^2 t + 16\sin^2 t)^{\frac{3}{2}}} \\ &= \frac{1}{(1 + 3\sin^2 t)^{\frac{3}{2}}}. \qquad [2 \text{ points}]\end{aligned}$$

Observing that

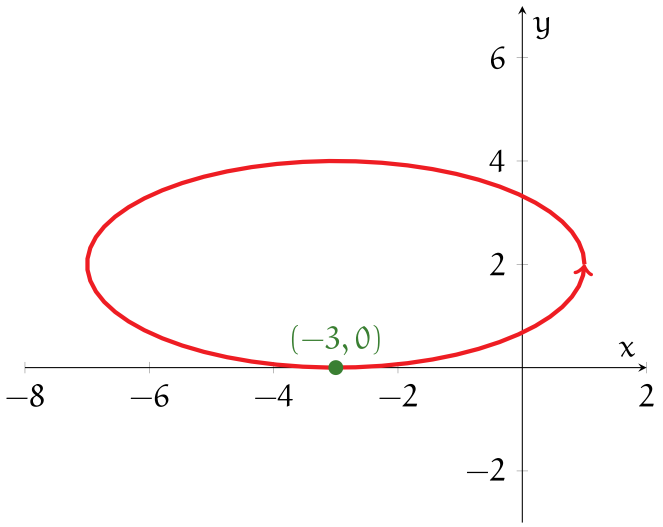
$$(-3, 0) = \gamma\left(-\frac{\pi}{2}\right).$$

then by definition, the curvature at $(-3, 0)$ is

$$\kappa_C(-3, 0) = \kappa_\gamma\left(-\frac{\pi}{2}\right) = \frac{1}{(1+3)^{\frac{3}{2}}} = \frac{1}{8}. \qquad [2 \text{ points}]$$

(b) [Seen similar] γ is drawn in red [4 points]; the point $(-3, 0)$ is in green [1 point]:

(Here, one needs not be so exact—getting the general shape of γ along with the positions of a few key values of γ would suffice.)



(c) [Unseen] We begin with the formula for the curvature of γ:

$$\kappa_\gamma(t) = \frac{1}{(1+3\sin^2 t)^{\frac{3}{2}}}.$$

Note that κ_γ is maximised when $\sin^2 t = 0$; this occurs when $t = 0, \pi$ [2 points].

As as result, the curvature of C is maximised at the points

$$\gamma(0) = (1, 2), \qquad \gamma(\pi) = (-7, 2). \quad \text{[2 point]}$$

(d) [Seen similar] The first step is to obtain a suitable parametrisation for integration. For this, we reduce the domain so that it is injective and still covers (almost) all of C:

$$\lambda : (0, 2\pi) \to \mathbb{R}^2, \qquad \lambda(t) = (-3 + 4\cos t, 2 + 2\sin t).$$

Note, however, that λ still has the wrong orientation [3 points].

Next, we compute the necessary quantities:

$$\mathbf{F}(\lambda(t)) = (-2 - 2\sin t, -3 + 4\cos t)_{(-3+4\cos t, 2+2\sin t)},$$
$$\lambda'(t) = (-4\sin t, 2\cos t). \quad \text{[2 point]}$$

Since λ has the wrong orientation, then:

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{ds} &= -\int_0^{2\pi} [(-2 - 2\sin t, -3 + 4\cos t) \cdot (-4\sin t, 2\cos t)] dt \\
&= -\int_0^{2\pi} (8\sin t + 8\sin^2 t - 6\cos t + 8\cos^2 t) dt \\
&= -\int_0^{2\pi} (8 + 8\sin t - 6\cos t) dt \\
&= -16\pi. \quad \text{[2 points]}
\end{aligned}$$

Question 2. [14 marks]

(a) Compute the tangent line at $t = 0$ to the parametric trefoil knot:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\sin t + 2\sin(2t), \cos t - 2\cos(2t), -\sin(3t)).$$ **[5]**

(b) Determine whether the following parametric curve is regular:

$$\alpha : \mathbb{R} \to \mathbb{R}^2, \qquad \alpha(t) = ((t-1)^3, (t-1)^2).$$

Justify your answer. **[5]**

(c) Give a parametrisation of the curve,

$$Q = \{(x, y) \in \mathbb{R}^2 \mid x^4 + (y+2)^4 = 1\},$$

that passes through the point $(0, -1)$. **Be sure to specify its domain.** **[4]**

(a) [Seen similar] Differentiating γ yields

$$\gamma'(t) = (\cos t + 4\cos(2t), -\sin t + 4\sin(2t), -3\cos(3t)).$$

In particular, at $t = 0$, we have that

$$\gamma(0) = (0, -1, 0), \qquad \gamma'(0) = (5, 0, -3). \quad \text{[3 points]}$$

Thus, by definition,

$$\begin{aligned} T_\gamma(0) &= \left\{ s \cdot \gamma'(0)_{\gamma(0)} \,\middle|\, s \in \mathbb{R} \right\} \\ &= \left\{ s \cdot (5, 0, -3)_{(0,-1,0)} \,\middle|\, s \in \mathbb{R} \right\}. \quad \text{[2 points]} \end{aligned}$$

(b) [Seen similar] Taking the derivative of α yields

$$\alpha'(t) = (3(t-1)^2, 2(t-1)). \quad \text{[1 point]}$$

Note in particular that $\alpha'(1) = (0, 0)$ (and hence $|\alpha'(1)| = 0$) [3 points].

As a result, α is not regular [1 point].

(c) [Seen similar] The most straightforward method is to set the parameter to be $t = x$. Using the defining equation for Q, we see that (for $t \in (-1, 1)$)

$$(y+2)^4 = 1 - x^4 = 1 - t^4, \qquad y = \pm(1-t^4)^{\frac{1}{4}} - 2.$$

To make sure the parametrisation passes through $(0, -1)$, we most choose "+" in the above. As a result, one correct parametrisation is given by

$$\lambda : (-1, 1) \to \mathbb{R}^2, \quad \text{[1 point]}$$
$$\lambda(t) = \left(t, (1-t^4)^{\frac{1}{4}} - 2\right). \quad \text{[3 points]}$$

(In particular, observe that $\lambda(0) = (0, -1)$.)

Question 3. [23 marks] Let S denote the surface of revolution given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 = y^2 + z^2,\ 0 < x < 2\}$$

and consider the following parametrisation of S:

$$\sigma : (0, 2) \times \mathbb{R} \to \mathbb{R}^3, \qquad \sigma(u, v) = (u, u^2 \cos v, u^2 \sin v).$$

(a) Compute the tangent plane to S at the point $(1, 0, 1)$. [5]

(b) Sketch the image of σ. On your sketch, draw (i) a path obtained by holding v constant and varying u, and (ii) a path obtained by holding u constant and varying v. [6]

(c) Find another parametrisation of S that generates the **opposite orientation** to the one generated by σ. **Be sure to specify its domain.** [4]

(d) Compute the surface integral

$$\iint_S H \, dA,$$

where H is the function

$$H : \mathbb{R}^3 \to \mathbb{R}, \qquad H(x, y, z) = \sqrt{1 + 4x^2}.$$ [8]

(a) [Seen similar] First, notice that

$$(1, 0, 1) = \sigma\left(1, \frac{\pi}{2}\right). \quad \text{[1 point]}$$

The partial derivatives of σ satisfy

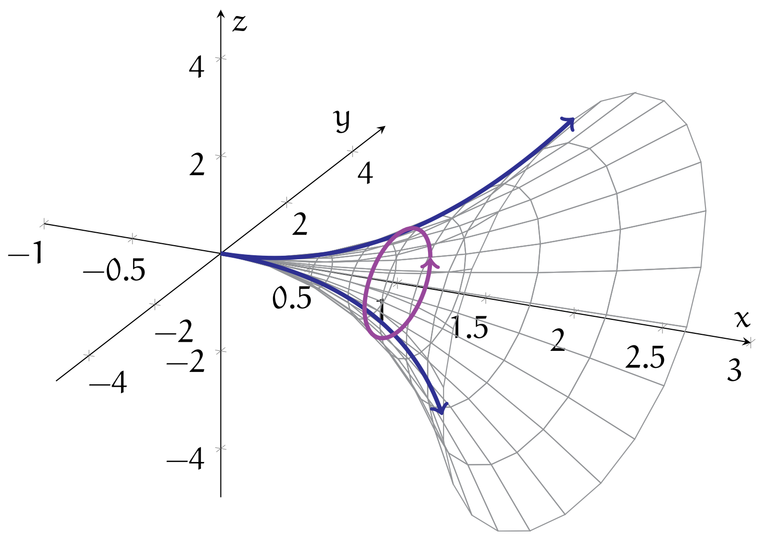
$$\partial_1 \sigma(u, v) = (1, 2u \cos v, 2u \sin v), \qquad \partial_2 \sigma(u, v) = (0, -u^2 \sin v, u^2 \cos v),$$
$$\partial_1 \sigma\left(1, \frac{\pi}{2}\right) = (1, 0, 2), \qquad \partial_2 \sigma\left(1, \frac{\pi}{2}\right) = (0, -1, 0). \quad \text{[3 points]}$$

Thus, by definition, the tangent plane is given by

$$T_{(1,0,1)} S = T_\sigma\left(1, \frac{\pi}{2}\right)$$
$$= \left\{a \cdot (1, 0, 2)_{(1,0,1)} + b \cdot (0, -1, 0)_{(1,0,1)} \,\middle|\, a, b \in \mathbb{R}\right\}. \quad \text{[1 point]}$$

(b) [Seen similar] The image of σ is drawn in the diagram below [4 points]; examples of level paths are drawn in (i) blue and (ii) purple, respectively [2 points].

(Here, one needs not be so exact—getting the general shape of σ along with the positions of a few key values of σ would suffice.)



(c) [Seen similar] This can be constructed by switching the roles of u and v:

$$\rho : \mathbb{R} \times (0, 2) \to \mathbb{R}^3, \quad \text{[1 point]}$$
$$\rho(u, v) = \sigma(v, u) = (v, v^2 \cos u, v^2 \sin u). \quad \text{[3 points]}$$

(d) [Seen similar] First, we parametrise by restricting the domain of σ:

$$\sigma : (0, 2) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \sigma(u, v) = (u, u^2 \cos v, u^2 \sin v).$$

Note that σ is now injective but still covers almost all of S [2 points].

Next, we collect the necessary intermediate computations:

$$\begin{aligned} H(\sigma(u, v)) &= \sqrt{1 + 4u^2}, \\ |\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)| &= u^2 |(2u, -\cos v, -\sin v)| \\ &= u^2 \sqrt{1 + 4u^2}. \quad \text{[3 points]} \end{aligned}$$

Combining the above, we compute that

$$\begin{aligned} \iint_S H \, dA &= \iint_{(0,2)\times(0,2\pi)} \sqrt{1 + 4u^2} \cdot u^2 \sqrt{1 + 4u^2} \, du dv \\ &= \int_0^{2\pi} dv \int_0^2 (u^2 + 4u^4) \, du \\ &= 2\pi \left(\frac{8}{3} + \frac{128}{5} \right). \quad \text{[3 points]} \end{aligned}$$

Question 4. [13 marks]

(a) Let $\mathbf{f}$ denote the following vector-valued function:

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{f}(x, y) = (xy^2, x^2y).$$

Find the directional derivative of $\mathbf{f}$ at the point $(1, 1)$ and in the direction $(-1, 2)$. [**4**]

(b) Explain (informally) why the surface integral of a real-valued function over a Möbius band well-defined, but the surface integral of a vector field over the same Möbius band is **not** well-defined. [**4**]

(c) Show that the following set is a surface:

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid x = y^3 + z^4\}.$$ [**5**]

(a) [Seen similar] Differentiating, we obtain that

$$\partial_1 \mathbf{f}(x, y) = (y^2, 2xy), \qquad \partial_2 \mathbf{f}(x, y) = (2xy, x^2),$$
$$\partial_1 \mathbf{f}(1, 1) = (1, 2), \qquad \partial_2 \mathbf{f}(1, 1) = (2, 1).$$ [2 points]

As a result, the directional derivative satisfies

$$d\mathbf{f}((-1, 2)_{(1,1)}) = -1 \cdot \partial_1 \mathbf{f}(1, 1) + 2 \cdot \partial_2 \mathbf{f}(1, 1)$$
$$= (3, 0).$$ [2 points]

(b) [Unseen] By definition, integrals of real-valued functions over surfaces are generally well-defined, but integrals of vector fields are only well-defined when the surface is assigned an orientation [2 points]. The Möbius band fails to be orientable [2 points].

(c) [Seen similar] Note that Z can be expressed as

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = 0\},$$

where h is the function defined as

$$h : \mathbb{R}^3 \to \mathbb{R}, \qquad h(x, y, z) = x - y^3 - z^4.$$ [2 point]

Since the gradient of h satisfies, for any $(x, y, z) \in \mathbb{R}^3$,

$$\nabla h(x, y, z) = (1, -3y^2, -4z^3)_{(x,y,z)} \neq (0, 0, 0)_{(x,y,z)},$$ [2 points]

it follows from the level set theorem that Z is indeed a surface [1 point].

Question 5. [15 marks] Using the method of Lagrange multipliers, find the maximum value of the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^2 + y^2,$$

subject to the constraint

$$(x-1)^2 + (y+1)^2 = 1.$$

Also, find all the points at which this maximum value is achieved. **[15]**

[Seen similar] Let h denote the function corresponding to the constraint:

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad h(x, y) = (x-1)^2 + (y+1)^2.$$

First, we compute the gradients of g and h:

$$\nabla g(x, y) = (2x, 2y)_{(x,y)}, \qquad \nabla h(x, y) = (2(x-1), 2(y+1))_{(x,y)}.$$

The method of Lagrange multipliers indicates that we must solve the system,

$$2x = \lambda \cdot 2(x-1), \qquad 2y = \lambda \cdot 2(y+1), \qquad (x-1)^2 + (y+1)^2 = 1.$$ [4 points]

We begin with the first two equations in the system. First, if $\lambda = 0$, then we have $x = y = 0$, which contradicts the constraint. Thus, it follows that $\lambda \neq 0$, and hence

$$1 + \frac{1}{y} = \frac{y+1}{y} = \frac{1}{\lambda} = \frac{x-1}{x} = 1 - \frac{1}{x}.$$

Rearranging the above, we conclude that

$$y = -x.$$

Plugging the above into the constraint yields $2(x-1)^2 = 1$, and it follows that

$$x = 1 \pm \frac{1}{\sqrt{2}}.$$

Since $y = -x$, then the maximum could only be achieved at one of the following:

$$\left(1 + \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}\right), \qquad \left(1 - \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}\right).$$ [7 points]

It remains only to check by applying g to the above points:

$$g\left(1 + \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}\right) = 3 + 2\sqrt{2},$$

$$g\left(1 - \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}\right) = 3 - 2\sqrt{2}.$$

Thus, the maximum value is given by

$$g_{\max} = 3 + 2\sqrt{2},$$ [2 points]

and $g_{\max}$ is achieved at the point

$$(x_{\max}, y_{\max}) = \left(1 + \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}\right).$$ [2 points]

Question 6. [13 marks]

(a) Let C be the circle centred at the origin and having radius 2,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\},$$

with the **anticlockwise** orientation. Use **Green's theorem** to compute

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (xe^{x^2} \ln(1 + x^2) - 3y, 3x + y^{18} \sinh y \cos y^2)_{(x,y)}.$$

(You may use that the area of the inside of a circle with radius R is πR^2.) [8]

(b) Bob wants to integrate a vector field $\mathbf{F}$ (on $\mathbb{R}^3$) over a cylinder of height 1,

$$S = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ 0 < z < 1\}.$$

Applying the **divergence theorem**, Bob obtains

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \iint_V (\nabla \cdot \mathbf{F})\, dx\, dy\, dz,$$

where V is the interior of the cylinder,

$$V = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 < 1,\ 0 < z < 1\}.$$

As a tutor for **MTH5113**, you decide that Bob must lose some marks for this. Where did Bob make a mistake? [5]

(a) [Seen similar] First, observe that $\mathcal{C}$ is the boundary of the region

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\},$$

i.e. the disk of radius 2 about the origin [2 points]. Also, note that $\mathbf{F}$ satisfies

$$\partial_x(3x + y^{18} \sinh y \cos y^2) - \partial_y(xe^{x^2} \ln(1 + x^2) - 3y) = 6. \quad [2 \text{ points}]$$

Thus, applying Green's theorem, we conclude that

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \iint_D \left[\partial_x(3x + y^{18} \sinh y \cos y^2) - \partial_y(xe^{x^2} \ln(1 + x^2) - 3y)\right] dx\, dy \\ &= \iint_D 6\, dx\, dy. \quad [3 \text{ points}]\end{aligned}$$

Finally, since the area of D is $\pi \cdot 2^2 = 4\pi$, then we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{ds} = 24\pi. \quad [1 \text{ point}]$$

(b) [Unseen] Bob applied the divergence theorem incorrectly—the boundary of V is not only the cylinder S, but also the two disks comprising the "top" and the "bottom" of V. Thus, the integrals over these disks must also be included [5 points].

End of Paper.