Goal: Higher-dimensional versions of fundamental theorem of calculus (FTC).

$$\underset{\text{2-d region}}{\iint} (\text{"derivative" of } F) \overset{?}{=} \underset{\text{1-d boundary}}{\int} F$$

~ What is the precise statement?

Thm: (Green's theorem) (1828, George Green) Let $D \subseteq \mathbb{R}^2$ be open and bounded, and assume the boundary of D consists of a union of curves $C_1, C_2, \ldots, C_k$. Also, let F be a smooth vector field on D and its boundary, and write the values of F as

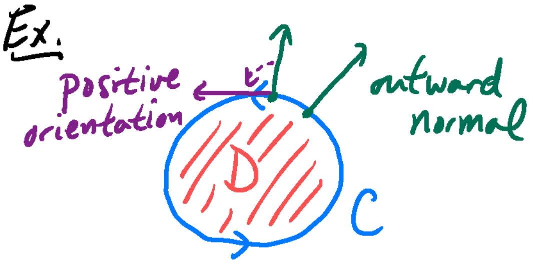
$$F(p) = (F_1(p), F_2(p))_p.$$

Then:
$$\underbrace{\iint_D}_{\text{integral over 2-d region}} \underbrace{[\partial_1 F_2(x,y) - \partial_2 F_1(x,y)]}_{\text{"derivative" of } F}\, dx\,dy = \underbrace{\int_{C_1} F \cdot d\vec{s} + \cdots + \int_{C_k} F \cdot d\vec{s}}_{\text{integral over 1-d boundary}},$$

where $C_1, \ldots, C_k$ are all given the "positive" orientation.

"turn left 90° from outward normal"

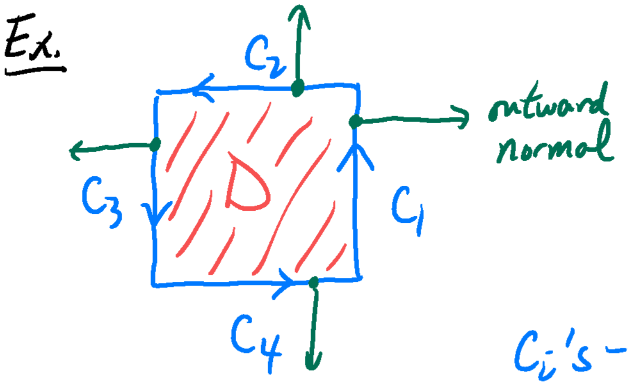
Ex.



D - disk C - circle (anticlockwise)

Green's theorem gives:
$$\iint_D [\partial_1 F_2(x,y) - \partial_2 F_1(x,y)]\,dx\,dy = \int_C F \cdot d\vec{s}$$
$$(F(p) = (F_1(p), F_2(p))_p)$$

Ex.

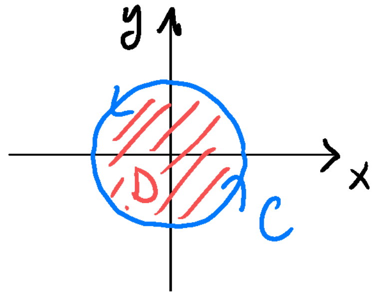


Green's theorem:
$$\iint_D [\partial_1 F_2(x,y) - \partial_2 F_1(x,y)]\,dx\,dy = \int_{C_1} F \cdot d\vec{s} + \cdots + \int_{C_4} F \cdot d\vec{s}$$

C_i's - oriented anticlockwise around D.

Ex. (Back to first example)

More concretely: $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$



$C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$
– anticlockwise orientation

F – vector field on $\mathbb{R}^2$
$F(x,y) = (\underbrace{x^2}_{F_1}, \underbrace{y^4}_{F_2})_{(x,y)}$

Find: $\int_C F \cdot d\vec{s}$

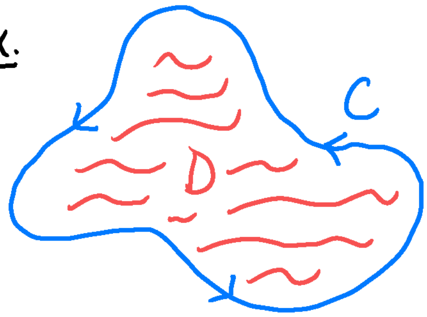
Method 1: Compute $\int_C F \cdot d\vec{s}$ directly (as before).

Method 2: Apply Green's theorem

$$\int_C F \cdot d\vec{s} = \iint_D \underbrace{[\partial_1 F_2(x,y) - \partial_2 F_1(x,y)]}_{\partial_x(y^4) - \partial_y(x^2) = 0} \, dx\,dy = 0$$

Remark: Though curve integral is zero, F is not normal to C.
– Subtle cancellations in $\int_C F \cdot d\vec{s}$ (easy to see from Green's theorem)

Ex.



Suppose D is (the surface of) a lake.
Goal: Find area of D.
Difficulty: you have no boat
– Can you measure area without venturing into the lake?

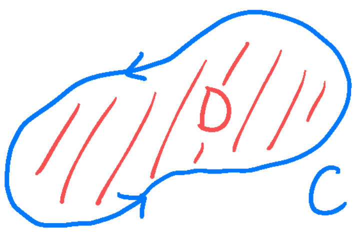
Consider vector field A on $\mathbb{R}^2$: $A(x,y) = (\underbrace{-\tfrac{1}{2}y}_{A_1}, \underbrace{\tfrac{1}{2}x}_{A_2})_{(x,y)}$

$\Longrightarrow \partial_1 A_2(x,y) - \partial_2 A_1(x,y) = \partial_x(\tfrac{1}{2}x) - \partial_y(-\tfrac{1}{2}y) = 1$

Thus: $A(D) = \iint_D 1 \, dA \underset{\text{Green}}{=} \int_C F \cdot d\vec{s}$

(*) Can measure $A(D)$ only by using measurements along C!

Remark: (Complex variables) Contour integrals can be formulated as curve integrals of vector fields.



Cauchy theorem: $\int_C f(z)\,dz = 0$
(f analytic)

– Can be proved using Green's theorem

Q1. What happens if D in Green's theorem is replaced by a surface?

(– But first, a detour.)

Def. Let $U \subseteq \mathbb{R}^3$ be open and connected, and let F be a smooth vector field on U, with $F(p) = (F_1(p), F_2(p), F_3(p))_p$.
We then define the curl of F, denoted $\nabla \times F$ or curl F, to be the vector field on U given by

$$(\nabla\times F)(p) = \underbrace{(\partial_2 F_3(p) - \partial_3 F_2(p),\ \partial_3 F_1(p) - \partial_1 F_3(p),\ \partial_1 F_2(p) - \partial_2 F_1(p))}_{= \text{"}\det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{bmatrix}\text{"}}{}_p$$

Q2. What does $\nabla\times F$ mean intuitively?

(*) In fact, Q1 and Q2 closely connected.

Ex. F - vector field on $\mathbb{R}^3$, $F(x,y,z) = (\underbrace{xy}_{F_1(x,y,z)}, \underbrace{x^2+yz}_{F_2(x,y,z)}, \underbrace{z^4+xyz}_{F_3(x,y,z)})_{(x,y,z)}$

- $\partial_2 F_3(x,y,z) - \partial_3 F_2(x,y,z) = \partial_y(z^4+xyz) - \partial_z(x^2+yz) = xz - y$
- $\partial_3 F_1(x,y,z) - \partial_1 F_3(x,y,z) = \partial_z(xy) - \partial_x(z^4+xyz) = 0 - yz = -yz$
- $\partial_1 F_2(x,y,z) - \partial_2 F_1(x,y,z) = \partial_x(x^2+yz) - \partial_y(xy) = 2x - x = x$

Thus $(\nabla\times F)(p) = (xz-y, -yz, x)_{(x,y,z)}$.

Answer to both Q1 and Q2 – Stokes' theorem

Thm. (Stokes' theorem) [1850s, George Stokes, Lord Kelvin]
Let $S \subseteq \mathbb{R}^3$ be a bounded, oriented surface, and assume the boundary of S is given by curves $C_1, C_2, \ldots, C_k$. Let F be a smooth vector field, defined on S and its boundary. Then:

$$\underbrace{\iint_S}_{\text{2-d integral}} \underbrace{(\nabla\times F)}_{\text{"derivative"}} \cdot d\vec{A} = \underbrace{\int_{C_1} F\cdot d\vec{s} + \ldots + \int_{C_k} F\cdot d\vec{s}}_{\text{1-d integral, over boundary of } S}$$

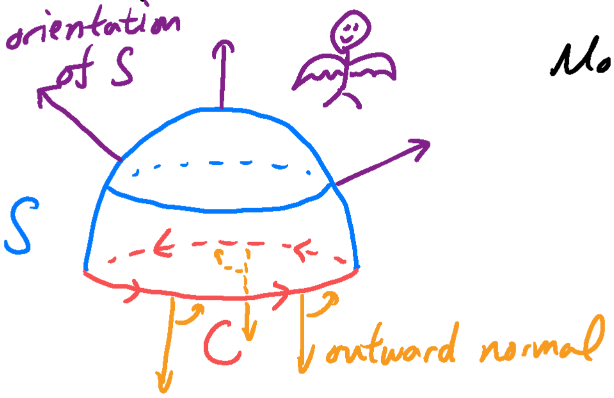
integral using orientation of S

integrals using "positive" orientation

Here $C_1, \dots, C_k$ are given the <u>positive orientation</u> with respect to the chosen orientation of S.

"Direction to the left from the outward normal to S, as viewed from the side of S given by its orientation."

<u>Ex.</u> S - upper half sphere, with "upward orientation"



More concretely:

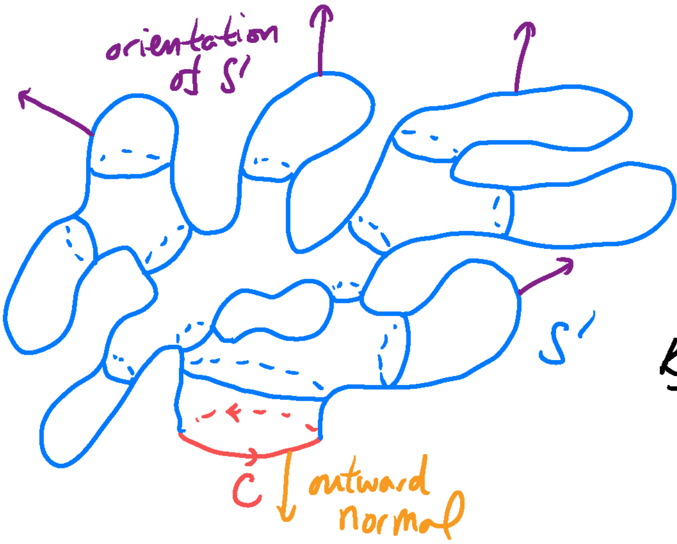
$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ z > 0\}$

$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ z = 0\}$

$= \{(x,y,0) \mid x^2+y^2 = 1\}$ - boundary of S

$\Rightarrow$ Stokes' theorem: $\underbrace{\iint_S (\nabla \times F) \cdot d\vec{A}}_{\text{upward orientation}} = \underbrace{\int_C F \cdot d\vec{s}}_{\text{orientation}}$

<u>Ex.</u> C - as before S' - drawn below (with "outward-facing orientation)



Consider vector field

$G(x,y,z) = (xz - y, -yz, x)_{(x,y,z)}$

<u>Find</u>: $\iint_{S'} G \cdot d\vec{A}$ (omg!!)

<u>Recall</u> (previous example): $G = \nabla \times F$

$F(x,y,z) = (xy, x^2+yz, z^4+xyz)_{(x,y,z)}$

<u>Stokes' theorem</u>: $\iint_{S'} G \cdot d\vec{A} = \iint_{S'} (\nabla \times F) \cdot d\vec{A} = \int_C F \cdot d\vec{s}$) much easier!

orientation

(Also Stokes' theorem: $\iint_S G \cdot d\vec{A} = \int_C F \cdot d\vec{s}$.)

For completeness, let's compute all 3 integrals:

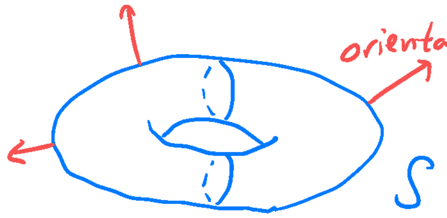
- Parametrisation of C: $\gamma : (0, 2\pi) \to C$, $\gamma(t) = (\cos t, \sin t, 0)$
  - ~ injective, covers "almost all" of C
  - ~ generates positive orientation of C

$$\Rightarrow \int_C F\cdot d\vec{s} = +\int_0^{2\pi} [\underbrace{F(\gamma(t))\cdot \gamma'(t)_{\gamma(t)}}_{(\cos t\sin t,\ \cos^2 t,\ 0)\cdot(-\sin t,\ \cos t,\ 0)}]\,dt$$

$$= \int_0^{2\pi} (-\cos t\sin^2 t + \underbrace{\cos^3 t}_{\cos t - \sin^2 t\cos t})\,dt = \int_0^{2\pi} (\cos t - 2\sin^2 t\cos t)\,dt$$

$$= \left[\sin t - \tfrac{2}{3}\sin^3 t\right]_{t=0}^{t=2\pi} = 0$$

Ex. S – torus (with "outward-facing" orientation)



- S has no boundary!

$\Rightarrow \iint_S (\nabla\times F)\cdot d\vec{A} = 0$ for any appropriate F.

Interpretation of curl – what does $\nabla\times F$ mean?

S – any surface passing through p, with boundary C.

Stokes' theorem: $\underbrace{\iint_S (\nabla\times F)\cdot d\vec{A}}_{\text{flux of } \nabla\times F \text{ through } S}$ = $\underbrace{\int_C F\cdot d\vec{s}}_{\text{how much } F \text{ points along } C}$

flux of $\nabla\times F$ through S " measures direction of $\nabla\times F$ near p

how much F points along C " measures how much F "circles around" p, along C.

Roughly: $(\nabla\times F)(p)$ quantifies how, and how much, F "circles around p" nearby p.

~ Different components of $(\nabla\times F)(p)$ capture different ways to go around C.

Next: go one dimension higher

$$\iiint_{\text{3-d region}} \text{"derivative of } F\text{"} \overset{?}{=} \iint_{\text{2-d boundary}} F \qquad ??$$

Q1: What is the precise statement here?

(First, a digression.)

Def. Let $U \subseteq \mathbb{R}^n$ be open and connected. Let F be a vector field on U, written as $F(p) = (F_1(p), \ldots, F_n(p))_p$. We define the divergence of F to be the real-valued function on U given by

$$(\nabla\cdot F)(p) = \partial_1 F_1(p) + \partial_2 F_2(p) + \ldots + \partial_n F_n(p).$$

also written $(\operatorname{div} F)(p)$

Q2. What does $\nabla \cdot F$ mean intuitively?

Ex. F - vector field on $\mathbb{R}^3$, $F(x,y,z) = (\underbrace{xy}_{F_1}, \underbrace{x^2+yz}_{F_2}, \underbrace{z^4+xyz}_{F_3})_{(x,y,z)}$

- $\partial_1 F_1(x,y,z) = y$ • $\partial_2 F_2(x,y,z) = z$ • $\partial_3 F_3(x,y,z) = 4z^3 + xy$

Thus, $(\nabla \cdot F)(x,y,z) = \underline{y + z + 4z^3 + xy}$

Answer to both Q1 and Q2 - given by divergence theorem
(Gauss, 1813; Lagrange, 1762; others, 1820s)

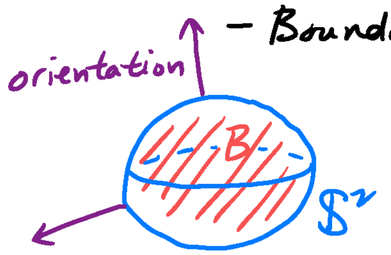
Thm (Divergence theorem) Let $W \subseteq \mathbb{R}^3$ be open and bounded, and assume the boundary of W is given by surfaces $S_1, S_2, \dots, S_k$. Also, let F be a smooth vector field on W and its boundary. Then,

$$\iiint_W (\nabla \cdot F)\, dV = \iint_{S_1} F \cdot d\vec{A} + \dots + \iint_{S_k} F \cdot d\vec{A},$$

where the S_i's are given the outward (from W) orientation.

Ex. Consider the unit ball $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$

- Boundary of $B = \mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$
  - given "outward-facing" orientation

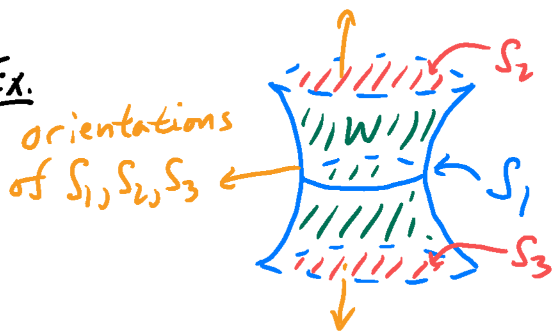


Consider vector field F on $\mathbb{R}^3$, given by
$$F(x,y,z) = (x,y,z)_{(x,y,z)}$$

(Previous example: $\iint_{\mathbb{S}^2} F \cdot d\vec{A} = 4\pi$
~ Let's recompute this using divergence theorem)

- $(\nabla \cdot F)(x,y,z) = \partial_x(x) + \partial_y(y) + \partial_z(z) = 3.$
- Divergence theorem $\Rightarrow$ $\iint_{\mathbb{S}^2} F \cdot d\vec{A} = \iiint_B \underbrace{(\nabla \cdot F)}_{3}\, dV = 3 \cdot \underbrace{\text{Volume}(B)}_{\frac{4\pi}{3}} = \underline{4\pi}$

Ex.



S_1 = side surface ("squeezed cylinder")
S_2 = top "lid"
S_3 = bottom "lid"
W = interior region bounded by S_1, S_2, S_3.

Then, for a smooth vector field F,

$$\iiint_W (\nabla\cdot F)\,dV = \underbrace{\iint_{S_1} F\cdot d\vec{A} + \iint_{S_2} F\cdot d\vec{A} + \iint_{S_3} F\cdot d\vec{A}}_{\text{with respect to outward orientation.}}.$$

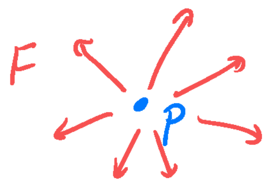
Interpretation of divergence - what does $\nabla\cdot F$ mean?

S - any surface enclosing p (outward orientation)
W - interior of S.

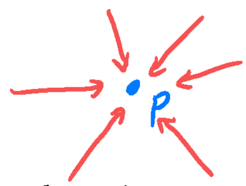
$$\Longrightarrow \underbrace{\iiint_W (\nabla\cdot F)\,dV}_{\text{measures } \nabla\cdot F \text{ near } p} = \underbrace{\iint_S F\cdot d\vec{A}}_{\text{how much } F \text{ is leaving } S}$$

how much F is leaving S
" how much F is pointing away from p.

Roughly: $(\nabla\cdot F)(p)$ quantifies how much F is pointing away from p nearby p.



$(\nabla\cdot F)(p) > 0$



$(\nabla\cdot F)(p) < 0$

Remark: The divergence theorem can be directly generalised to all dimensions.

$$\int_{\underbrace{W}} \underbrace{(\nabla\cdot F)}_{\text{divergence}} = \int_{\underbrace{\partial W}} (F\cdot \underbrace{N}_{\text{outward unit normal from } W})$$

$W \subseteq \mathbb{R}^n$ - open, bounded n-dimensional region

boundary of W
$\sim (n-1)$-dimensional geometric object

In particular: $n=1 \Rightarrow$ becomes fundamental theorem of calculus

$n=2 \Rightarrow$ becomes Green's theorem

There is a more general result - "generalised Stokes' theorem"
- includes FTC, Green's theorem, Stokes' theorem, divergence theorem

- Usually written as: $\int_M d\alpha = \int_{\partial M} \alpha$
    - M – n-dimensional geometric object (manifold)
    - ∂M – boundary of M
    - α – "differential form" (special type of object)
    - $d\alpha$ – "exterior derivative" of α (special derivative operation)

~ For example:

$$\underbrace{n=2 \Rightarrow \begin{cases} \bullet\ M \text{ is a surface} \\ \bullet\ \alpha \simeq \text{vector field} \\ \bullet\ d\alpha \simeq \text{curl} \end{cases}}_{\text{Stokes' theorem}}$$

$$\underbrace{n=1 \Rightarrow \begin{cases} \bullet\ M \text{ is a curve} \\ \bullet\ \alpha \simeq \text{scalar function} \\ \bullet\ d\alpha \simeq \text{gradient} \end{cases}}_{\text{FTC for curves.}}$$