Constrained Optimisation Problems

(P1) Extremise $f(x,y)$, subject to constraint $g(x,y)=c$.

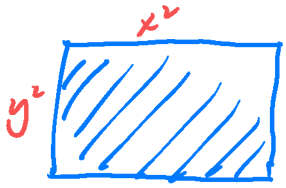
Geometric problem: (P1*) Extremise $f(x,y)$ on curve $C=\{(x,y) \mid g(x,y)=c\}$

Recall: If f achieves extremum at $p \in C$, then

$$\nabla f(p) = \lambda \cdot \nabla g(p) \text{ for some } \lambda \in \mathbb{R}.$$

$\Longrightarrow$ Method of Lagrange multipliers.

Ex. Given material to build a 160m. fence, what is the largest rectangular area you can enclose?



(*) Maximise $f(x,y)=x^2y^2$, subject to constraint $g(x,y)=x^2+y^2=80$

$(2x^2+2y^2=160)$

Step 0: Derive system of equations

$$\begin{pmatrix} f: \mathbb{R}^2 \to \mathbb{R} \\ g: \mathbb{R}^2 \to \mathbb{R} \end{pmatrix} \quad \begin{array}{ll} \partial_1 f(x,y) = 2y^2x & \partial_2 f(x,y) = 2x^2y \\ \partial_1 g(x,y) = 2x & \partial_2 g(x,y) = 2y \end{array}$$

$\Longrightarrow$ system: $\overset{(A)}{2y^2x=\lambda\cdot 2x}$ $\quad$ $\overset{(B)}{2x^2y=\lambda\cdot 2y}$ $\quad$ $\overset{(C)}{x^2+y^2=80}$

$\nabla f = \lambda \cdot \nabla g$ $\quad$ constraint

Step 1: Solve system for (x,y,λ) $\sim$ here, split in cases.

- If $\lambda=0$: (A) trivially holds
  (C) $\Rightarrow y^2=80 \Rightarrow y=\pm\sqrt{80}$
  (B) $\Rightarrow \lambda\cdot 2(\pm\sqrt{80}) = 2x^2y=0 \Rightarrow \lambda=0$

  $\Longrightarrow$ 2 solutions: $(x,y,\lambda)=(0,\pm\sqrt{80},0)$

- If $y=0$: (B) trivially holds
  (C) $\Rightarrow x^2=80 \Rightarrow x=\pm\sqrt{80}$
  (A) $\Rightarrow \lambda\cdot 2(\pm\sqrt{80})=0 \Rightarrow \lambda=0$

  $\Longrightarrow$ 2 solutions: $(x,y,\lambda)=(\pm\sqrt{80},0,0)$

- If $x\neq 0$ and $y \neq 0$: Can divide by x and y

  (A), (B) $\Longrightarrow$ $\underbrace{y^2 = \frac{2y^2x}{2x} = \lambda = \frac{2x^2y}{2y} = x^2}$

(C) $\Rightarrow 2x^2 = 80 \Rightarrow x = \pm\sqrt{40} \Rightarrow y = \pm\sqrt{40},\ \lambda = 40$

$\Rightarrow$ 4 solutions: $(x,y,\lambda) = (\pm\sqrt{40}, \pm\sqrt{40}, 40)$

(*) If maximum of f is achieved on C, then it must be at one of these solution points (x,y).

Step 2: Check each point.

- $f(0, \pm\sqrt{80}) = 0^2 \cdot 80 = 0$ (no)
- $f(\pm\sqrt{80}, 0) = 80 \cdot 0^2 = 0$ (no)
- $f(\pm\sqrt{40}, \pm\sqrt{40}) = 40 \cdot 40 = 1600$ (maybe)

(*) 1600 may be maximum value of f.
(also possible that maximum does not exist!)

Remark: In fact, 1600 is the maximum (will show this later)

$\Rightarrow$ max enclosed area: 1600 (m^2)

- achieved when $x^2 = y^2 = 40$ (square!)

40

40 

Constrained Optimisation in 3 dimensions

(P2) Extremise $f(x,y,z)$, subject to constraint $g(x,y,z) = c$.

Setup:
- $U \subseteq \mathbb{R}^3$ – open, connected
- $f: U \to \mathbb{R}$, $g: U \to \mathbb{R}$ – smooth functions
- $S = \{(x,y,z) \in U \mid g(x,y,z) = c\}$
  - Assume $\nabla g(p) \neq \vec{0}_p$ for all $p \in S$ ($\Rightarrow S$ is a surface)

Thus, can reformulate (P2) as:

(P2*) Extremise f on the surface S.

Thm: Assume the above setup.
Suppose f achieves its extremum value on S at $p \in S$. Then:
- $\nabla f(p)$ is normal to every element of T_pS.
- There exists $\lambda \in \mathbb{R}$ such that $\nabla f(p) = \lambda \cdot \nabla g(p)$

(Proof analogous to that for 2-d problem.)

Again, leads to method of Lagrange multipliers:

- <u>Step 1</u>: Solve system

$$\begin{aligned} \partial_1 f(x,y,z) &= \lambda\, \partial_1 g(x,y,z) \\ \partial_2 f(x,y,z) &= \lambda\, \partial_2 g(x,y,z) \\ \partial_3 f(x,y,z) &= \lambda\, \partial_3 g(x,y,z) \end{aligned} \Bigg\} \nabla f(p) = \lambda \cdot \nabla g(p)$$

$$g(x,y,z) = c \;\; \} \; p \in S$$

for unknowns (x,y,z,λ).

- <u>Step 2</u>: For each solution (x,y,z,λ), check if f achieves maximum or minimum at (x,y,z).

<u>Ex.</u> Find the maximum and minimum values of $4xy+z$, subject to the constraint $x^2+y^2+z^2=1$. At which points are the maximum/minimum values achieved?

- <u>Setting</u>: $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 4xy+z$ – optimise this
  $g: \mathbb{R}^3 \to \mathbb{R}$, $g(x,y,z) = x^2+y^2+z^2 = 1$ – Constraint

$$\partial_1 f(x,y,z) = 4y \qquad \partial_2 f(x,y,z) = 4x \qquad \partial_3 f(x,y,z) = 1$$
$$\partial_1 g(x,y,z) = 2x \qquad \partial_2 g(x,y,z) = 2y \qquad \partial_3 g(x,y,z) = 2z$$

$\Rightarrow$ system: (to solve) $\underset{(A)}{4y = \lambda \cdot 2x}$ $\quad \underset{(B)}{4x = \lambda \cdot 2y}$ $\quad \underset{(C)}{1 = \lambda \cdot 2z}$ $\quad \underset{(D)}{x^2+y^2+z^2=1}$

<u>Cases</u>: • If <u>$x=0$</u>: $(A) \Rightarrow y=0$, (B) trivially holds
$(D) \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$, $(C) \Rightarrow \lambda = \frac{1}{2z}$

$\Rightarrow$ Solutions: $(x,y,z,\lambda) = (0,0,+1,+\frac{1}{2}),\ (0,0,-1,-\frac{1}{2})$.

values: $f(0,0,+1) = +1$, $f(0,0,-1) = -1$

• If <u>$y=0$</u>: $(B) \Rightarrow x=0$, (A) trivially holds
$(D) \Rightarrow z = \pm 1$, $(C) \Rightarrow \lambda = \frac{1}{2z}$

(Same solutions as $x=0$ case)

• If $x \neq 0$ and $y \neq 0$: $(A),(B) \Rightarrow \frac{2y}{x} = \lambda = \frac{2x}{y}$ (divide by x, y)

$\Rightarrow x^2 = y^2 \Rightarrow x = \pm y$ {also, $\lambda = \frac{2x}{y}$ $(= \pm 2)$

(C) $z = \frac{1}{2\lambda}$ $(= \pm \frac{1}{4})$

$(D) \Rightarrow 2x^2 + \frac{1}{16} = 1 \Rightarrow x = \pm\sqrt{\frac{15}{32}}$, $y = \pm\sqrt{\frac{15}{32}}$

$\Longrightarrow$ 4 solutions: (x, y, z, λ)

$\left(+\sqrt{\tfrac{15}{32}}, +\sqrt{\tfrac{15}{32}}, +\tfrac{1}{4}, +2\right)$ $f = +\tfrac{17}{8}$

$\left(+\sqrt{\tfrac{15}{32}}, -\sqrt{\tfrac{15}{32}}, -\tfrac{1}{4}, -2\right)$ $f = -\tfrac{17}{8}$

$\left(-\sqrt{\tfrac{15}{32}}, +\sqrt{\tfrac{15}{32}}, -\tfrac{1}{4}, -2\right)$ $f = -\tfrac{17}{8}$

$\left(-\sqrt{\tfrac{15}{32}}, -\sqrt{\tfrac{15}{32}}, +\tfrac{1}{4}, +2\right)$ $f = +\tfrac{17}{8}$

Thus, possible maximum value is $+\tfrac{17}{8}$:
- achieved at $\left(+\sqrt{\tfrac{15}{32}}, +\sqrt{\tfrac{15}{32}}, +\tfrac{1}{4}\right), \left(-\sqrt{\tfrac{15}{32}}, -\sqrt{\tfrac{15}{32}}, +\tfrac{1}{4}\right)$

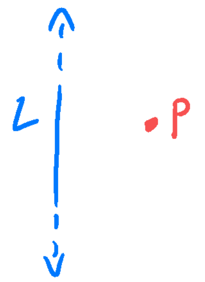
possible minimum value is $-\tfrac{17}{8}$:
- achieved at $\left(+\sqrt{\tfrac{15}{32}}, -\sqrt{\tfrac{15}{32}}, -\tfrac{1}{4}\right), \left(-\sqrt{\tfrac{15}{32}}, +\sqrt{\tfrac{15}{32}}, -\tfrac{1}{4}\right)$

min/max value may not exist!

Remark: In fact, these are the maximum/minimum values. (discussion upcoming)

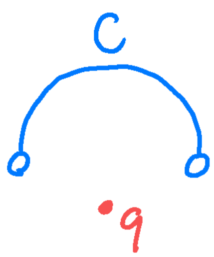
Q. In previous examples, how do you know extremum exists?

Ex. This is a serious issue.



f = distance to p

f has minimum on L, but no maximum



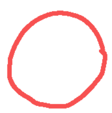
g = distance to q

g has maximum on C, but no minimum

Thm: Let M be a curve or a surface, and let f be a smooth real-valued function whose domain includes M. If M is closed and bounded, then f achieves both a maximum and a minimum on M.

- Bounded – "does not go off to infinity"

(curves)



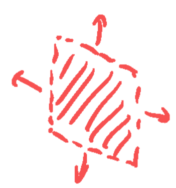
bounded

not bounded

(surfaces)

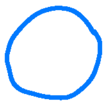
bounded

bounded



not bounded

- **Closed** – has no "boundary/edge points"

(curves)  closed  closed  not closed

(surfaces)  closed  not closed

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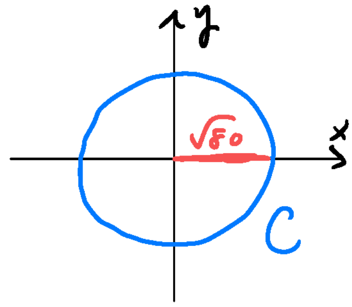
Ex. (Fence problem) Maximize $f(x,y) = x^2y^2$, subject to the constraint $g(x,y) = x^2+y^2 = 80$.

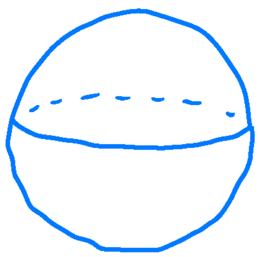


- Constraint curve: $C = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 80\}$
  – circle – closed and bounded
  $\Rightarrow f$ has maximum (and minimum) on C!
  ~ can be found via Lagrange multipliers

Recall: Solutions of system

- $f(0, \pm\sqrt{80}) = 0 = f(\pm\sqrt{80}, 0)$ must be minimum
- $f(\pm\sqrt{40}, \pm\sqrt{40}) = 1600$ must be maximum

Ex. Find extrema of $f(x,y,z) = 4xy + z$, subject to constraint $x^2+y^2+z^2 = 1$.



- Constraint surface: $S = S^2$ (closed, bounded)
  $\Rightarrow f$ has maximum and minimum on S.

Recall: From method of Lagrange multipliers:

- possible max: $+\frac{17}{8} = f\left(+\sqrt{\frac{15}{32}}, +\sqrt{\frac{15}{32}}, +\frac{1}{4}\right) = f\left(-\sqrt{\frac{15}{32}}, -\sqrt{\frac{15}{32}}, +\frac{1}{4}\right) \Rightarrow$ must be actual max
- possible min: $-\frac{17}{18} = f\left(+\sqrt{\frac{15}{32}}, -\sqrt{\frac{15}{32}}, -\frac{1}{4}\right) = f\left(-\sqrt{\frac{15}{32}}, +\sqrt{\frac{15}{32}}, -\frac{1}{4}\right) \Rightarrow$ must be actual min

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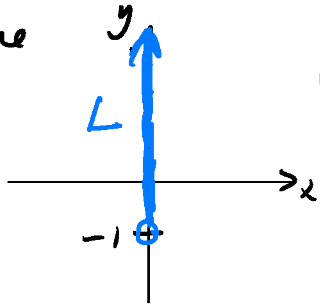
Q. What if curve/surface M is not closed and bounded?
– Must consider other possibilities.

(1) Extremum achieved at point of M (found via Lagrange multipliers)
(2) Approach extremum at boundary of M. \ extremum does
(3) Approach extremum "toward infinity" / not exist on M.

Ex. Find extrema of $f(x,y)=y^2$ on the half-line

$L=\{(x,y)\in\mathbb{R}^2 \mid \underbrace{x}_{g(x,y)}=0 \text{ and } y>-1\}$



- $g(x,y)=x$
- $\nabla f=\lambda\cdot\nabla g \Rightarrow 0=\lambda\cdot 1,\ 2y=\lambda\cdot 0,\ x=0$

$\Rightarrow$ only solution $(x,y,\lambda)=(0,0,0)$

(1) From Lagrange multipliers $\Rightarrow f(0,0)=0$)– minimum

(2) Toward boundary point $(0,-1) \Rightarrow f(0,-1)=1$)– neither, not in L

(3) Toward infinity $\Rightarrow f(0,y)\xrightarrow{y\to+\infty}+\infty$)– maximum, but not in L

(*) • f attains minimum value, 0, at $(0,0)\in L$.
• f does not attain maximum value on L.

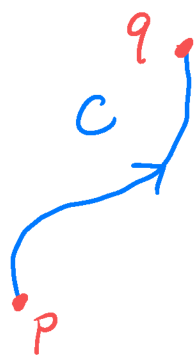
Recall: Fundamental theorem of Calculus (FTC)

$$\underbrace{\int_a^b \underbrace{f'(x)}_{\text{derivative}}\,dx}_{\text{integral over 1-d interval}} = \underbrace{f(b)-f(a)}_{\text{"integral" over 0-d boundary of } (a,b)}$$

Q. Extensions of the FTC?
- To geometric settings?
- In higher dimensions?

First task: extend FTC to curve integrals
(same dimension, but replace intervals by curves)

Thm: Let $C\subseteq\mathbb{R}^n$ be a bounded, oriented curve, with "initial" point $p\in\mathbb{R}^n$ and "final point" $q\in\mathbb{R}^n$. Then, for any smooth, real-valued function f whose domain contains C, p, q, we have



$$\underbrace{\int_C \underbrace{\nabla f}_{\text{"derivative"}}\cdot d\vec{s}}_{\text{integral over 1-d curve}} = \underbrace{f(q)-f(p)}_{\text{"integral" over 0-d boundary of } C.}$$

<u>Proof (sketch)</u>: There is an injective parametrisation $\gamma : (a,b) \to C$ of C such that:

- $\lim_{t \searrow a} \gamma(t) = p$ • $\lim_{t \nearrow b} \gamma(t) = q$ (γ generates our orientation of C)

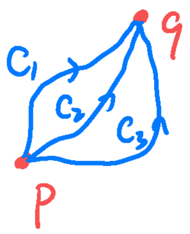
$$\Rightarrow \int_C \nabla f \cdot d\vec{s} = \int_a^b \underbrace{\nabla f(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}}_{\frac{d}{dt}[f(\gamma(t))] \text{ (chain rule)}} \, dt$$

$$\overset{\text{(FTC)}}{=} \lim_{t \nearrow b} f(\gamma(t)) - \lim_{t \searrow a} f(\gamma(t)) = f(q) - f(p).$$

<u>Observe</u>: $\underbrace{\int_C \nabla f \cdot d\vec{s}}_{\text{seems to involve values of } f \text{ over all of } C.} = \underbrace{f(q) - f(p)}_{\text{depends only on } f \text{ at } p \text{ and } q}$

(✱) $\int_C \nabla f \cdot d\vec{s}$ depends only on $f(p), f(q)$ – not on path C from p to q } – "independence of path"

↳ ∇f is a <u>conservative</u> vector field



$$\Rightarrow \int_{C_1} \nabla f \cdot d\vec{s} = \int_{C_2} \nabla f \cdot d\vec{s} = \int_{C_3} \nabla f \cdot d\vec{s} = f(q) - f(p)$$

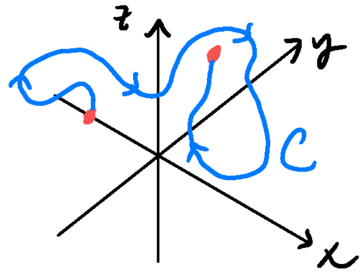
<u>Ex.</u> G – vector field on $\mathbb{R}^3 \setminus \{(0,0,0)\}$

$$G(p) = -\frac{1}{|p|^3} P_p \quad \text{(Newtonian gravitational force)}$$

<u>Recall</u>: $G = \nabla g$, where $g(p) = \frac{1}{|p|}$ (gravitational potential)

Let C be any bounded, oriented curve in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, with

- starting point $(-1,0,0)$
- ending point $(0,1,1)$



Work done by force $\vec{G}$ on object travelling along path C:

$$\int_C G \cdot d\vec{s} = \int_C \nabla g \cdot d\vec{s} = g(0,1,1) - g(-1,0,0)$$

$$= \frac{1}{\sqrt{2}} - 1$$

~ work done independent of path taken.

Q. What about in higher dimensions? What is the FTC?

$$\iint_{\text{2-d region } A} (\text{"derivative" of } F) \stackrel{?}{=} \int_{\substack{\text{1-d boundary} \\ \text{of } A}} F \quad ??$$

- First, will consider case where $A \subseteq \mathbb{R}^2$ $\Rightarrow$ $\boxed{\text{Green's}}$ theorem

George Green (baker, miller, self-taught mathematician)

first version 1828