<u>Last</u> <u>time</u>: Integrals over parametric surfaces.

$\sigma: U \to \mathbb{R}^3$ – parametric surface

$$\Rightarrow \iint_\sigma F \, dA = \iint_U F(\sigma(u,v)) \, |\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| \, du dv$$

<u>Q</u>. Are these integrals independent of parametrisation ("geometric")?
(<u>A</u>. yes)

<u>Thm</u>: Let $\sigma: U \to \mathbb{R}^3$ and $\tilde\sigma: \tilde U \to \mathbb{R}^3$ be regular parametric surfaces, and assume $\sigma, \tilde\sigma$ are reparametrisations of each other. Then, for any real-valued function F defined on the images of σ and $\tilde\sigma$,

$$\iint_\sigma F \, dA = \iint_{\tilde\sigma} F \, dA$$

- In particular, $A(\sigma) = A(\tilde\sigma)$.

<u>Proof ideas</u>: Calculus computation, using change of variables formula.

<u>Show</u>: $\iint_U F(\sigma(u,v)) \, |\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| \, du dv = \iint_{\tilde U} F(\tilde\sigma(\tilde u,\tilde v)) \, |\partial_1\tilde\sigma(\tilde u,\tilde v) \times \partial_2\tilde\sigma(\tilde u,\tilde v)| \, d\tilde u d\tilde v$

(*) Jacobian determinant transforms $|\partial_1\sigma \times \partial_2\sigma|$ to $|\partial_1\tilde\sigma \times \partial_2\tilde\sigma|$.

Thus, can now make sense of integrals over <u>surfaces</u>.

<u>Def</u>. Let $S \subseteq \mathbb{R}^3$ be a surface, and let F be a real-valued function defined on S. Then, we define the <u>surface integral</u> of F over S by

$$\iint_S F \, dA = \iint_\sigma F \, dA,$$

where σ is <u>any</u> injective parametrisation of S such that <u>S differs from the image of σ only by a finite collection of points and curves</u>.

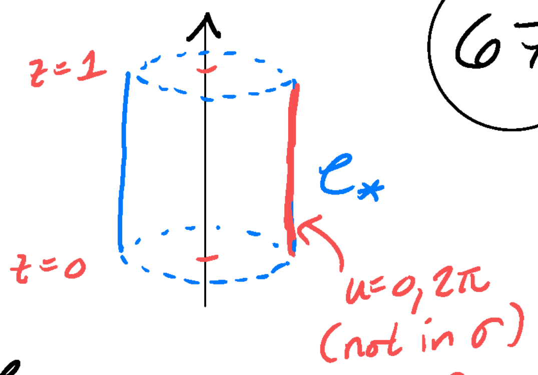
(surface area of S)

- Similarly, for the same setup, we define $A(S) = A(\sigma)$.

(*) Avoids counting points of S more than once.

(*) Individual points/curves should have zero area.

Ex. $\mathcal{C}_* = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1,\ 0<z<1\}$
(cylindrical segment)



Let $G: \mathbb{R}^3 \to \mathbb{R}$, $G(x,y,z) = x+y+z$.

- Find $\iint_{\mathcal{C}_*} G \, dA$.

Step 1: Find good parametrisation of $\mathcal{C}_*$.

$\sigma: \underbrace{(0,2\pi)}_{u}\times\underbrace{(0,1)}_{v} \to \mathcal{C}_*$, $\sigma(u,v) = (\cos u, \sin u, v)$

u: goes once around circles of $\mathcal{C}_*$

v: restricts z-values between 0 and 1.

- σ is injective
- Image of σ is all of $\mathcal{C}_*$, except one vertical line
(u=0, u=2π)

Step 2: Compute!

Recall: $|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| = |(\cos u, \sin u, 0)| = 1$.

Also: $G(\sigma(u,v)) = G(\cos u, \sin u, v) = \cos u + \sin u + v$.

Thus, $$\iint_{\mathcal{C}_*} G \, dA = \iint_{(0,2\pi)\times(0,1)} \underbrace{G(\sigma(u,v))}_{\cos u+\sin u+v} \underbrace{|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|}_{1} \, du\,dv$$

$$= \int_0^1 \underbrace{\int_0^{2\pi} (\cos u + \sin u + v)\,du}\,dv$$

$\to [\sin u - \cos u + vu]_{u=0}^{u=2\pi} = 2\pi v$

$$= \int_0^1 2\pi v \, dv = \pi.$$

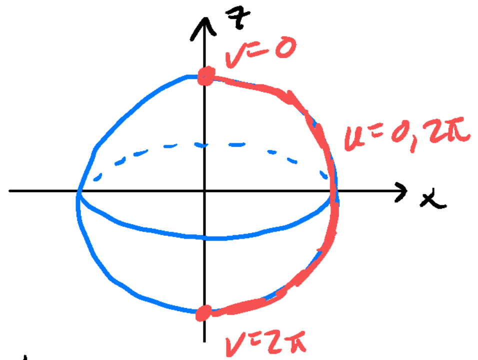
Ex. Find the surface area of $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

Again, find good parametrisation:

$p: (0,2\pi)\times(0,\pi) \to \mathbb{S}^2$ (spherical coordinates)

$p(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

~ injective, image covers all of $\mathbb{S}^2$ but



Then: $$A(\mathbb{S}^2) = A(p) = \int_0^{2\pi}\int_0^{\pi} \underbrace{|\partial_1 p(u,v) \times \partial_2 p(u,v)|}_{\sin v}\, du\,dv$$

$= 2\pi \int_0^\pi \sin v \, dv = \underline{4\pi}$ (previous example)

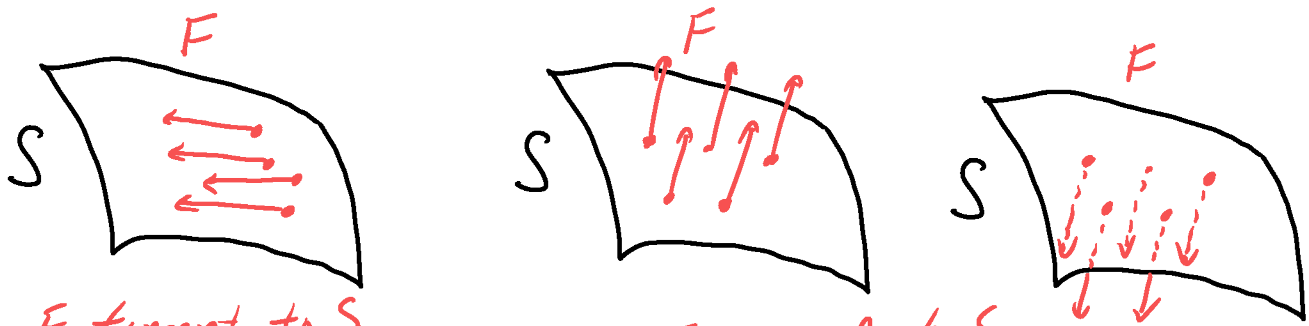
Remark: Also possible to define integrals over surfaces $S \subseteq \mathbb{R}^n$ (not just $\mathbb{R}^3$). (See lecture notes, exercises)

Next topic: Surface integrals over vector fields (rather than scalars)

Motivation (physics): flux – "rate of flow through a surface"

- "flow" – could be fluids, gravitational/electromagnetic force.

vector field F ~ $F(p)$ – velocity of fluid at position p.



F tangent to S (zero flux)

F normal to S (nonzero flux, but with opposite signs)

Ideas: • only measure component of F normal to S.

~ $F \cdot$ (unit normal) ~ dot product

• associate one direction through S with positive flux, other direction with negative flux.

(*) But, which direction?

– Need to make extra choice.

~ choice of direction with positive flux

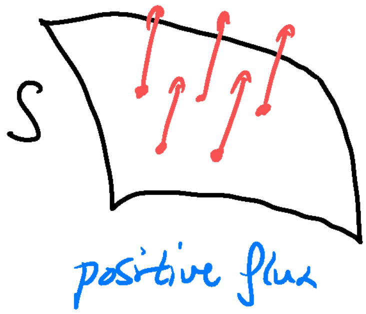
$\Longleftrightarrow$ choice of unit normal n_p at each $p \in S$.

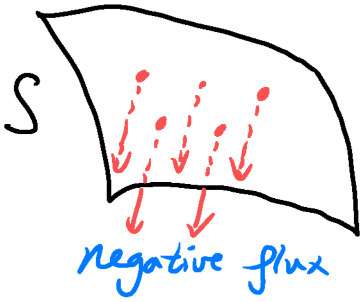
$\Longleftrightarrow$ choice of orientation of S

(chosen side = positive flux)

(*) Need an oriented surface to define flux.

Ex. Choose "upward-facing" orientation of S.





Def. Let $S \subseteq \mathbb{R}^3$ be an oriented surface, and let F be a vector field defined on S. Then, we define the <u>surface integral</u> of F over S by

$$\iint_S F \cdot d\vec{A} = \iint_S \underbrace{(F \cdot n)}_{\text{scalar}} \, dA \Big) \text{ integral of scalar function}$$

where n denotes the unit normals of S in the direction specified by the orientation of S.

Q. What happens when orientation of S is reversed?

- n replaced by $-n$
- Everything else does not depend on orientation (geometric)

(*) Thus, $\iint_S F \cdot d\vec{A}$ replaced by $-\iint_S F \cdot d\vec{A}$.

Q. How to compute surface integrals of vector fields?

(A. Via parametrisations!)

Thm. Let S, F be as before, and let $\sigma : U \to S$ be an injective parametrisation of S, whose image differs from S by a finite collection of points and curves.

- If σ generates the orientation of S, then

$$\iint_S F \cdot d\vec{A} = + \iint_U F(\sigma(u,v)) \cdot [\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)]_{\sigma(u,v)} \, du\, dv$$

- If σ generates an orientation opposite to that of S, then

$$\iint_S F\cdot d\vec{A} = -\iint_U F(\sigma(u,v))\cdot[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)}\,du\,dv$$

Proof: (Consider only the "+" case ~ "–" case is similar)

σ generates orientation of S

unit normal satisfies $n_{\sigma(u,v)} = +\left[\frac{w(u,v)}{|w(u,v)|}\right]_{\sigma(u,v)}$

$w(u,v) = \partial_1\sigma(u,v)\times\partial_2\sigma(u,v)$

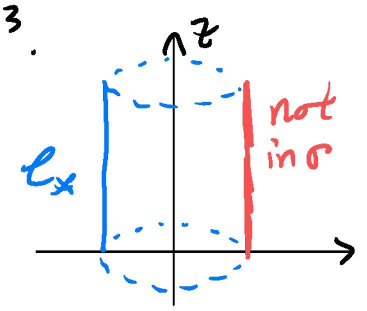
Thus:
$$\iint_S F\cdot d\vec{A} = \iint_S (F\cdot n)\,dA$$
$$= \iint_U \left(F(\sigma(u,v))\cdot\left[+\frac{w(u,v)}{|w(u,v)|}\right]_{\sigma(u,v)}\right)|w(u,v)|\,du\,dv$$
$$= \iint_U [F(\sigma(u,v))\cdot w(u,v)_{\sigma(u,v)}]\,du\,dv$$

Ex. $\mathcal{C}_* = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1,\ 0<z<1\}$ (cylinder segment)

– given "outward-facing" orientation.

$F(x,y,z) = (x,y,z)_{(x,y,z)}$ – vector field on $\mathbb{R}^3$.

- Find: $\iint_{\mathcal{C}_*} F\cdot d\vec{A}$



Step 1: Find good parametrisation

Recall: $\sigma: (0,2\pi)\times(0,1)\to\mathcal{C}_*$, $\sigma(u,v) = (\cos u, \sin u, v)$

- injective, image is "almost all" of $\mathcal{C}_*$.

Also: $[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)} = (\cos u, \sin u, 0)_{(\cos u, \sin u, v)}$

– points outward from $\mathcal{C}_*$ (can see from plotting)

$\Rightarrow$ σ generates outward orientation of $\mathcal{C}_*$.

~ σ matches our given orientation!

Step 2: Compute: $F(\sigma(u,v)) = (\cos u, \sin u, v)_{(\cos u, \sin u, v)}$

$$\iint_{\mathcal{C}^*} F\cdot d\vec{A} = +\iint_{(0,2\pi)\times(0,1)} F(\sigma(u,v))\cdot[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)}\,du\,dv$$

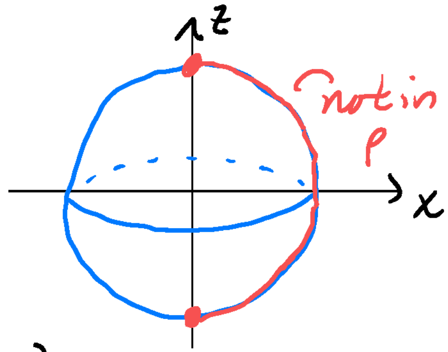
$$= \int_0^{2\pi}\Big[\int_0^1 \underbrace{(\cos u,\sin u,v)\cdot(\cos u,\sin u,0)}_{\cos^2 u+\sin^2 u+v\cdot 0=1}\,dv\Big]du$$

$$= \int_0^{2\pi}du\int_0^1 dv = 2\pi$$

Ex. $\mathbb{S}^2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ – sphere

– with "outward-facing" orientation

Find: $\iint_{\mathbb{S}^2} F\cdot d\vec{A}$, with F as before.



Parametrisation: $\rho: (0,2\pi)\times(0,\pi)\longrightarrow \mathbb{S}^2$

$\rho(u,v) = (\cos u\sin v, \sin u\sin v, \cos v)$

– injective, covers "almost all" of $\mathbb{S}^2$

Recall: $[\partial_1\rho(u,v)\times\partial_2\rho(u,v)]_{\rho(u,v)} = -\underbrace{\sin v}_{>0}\cdot\rho(u,v)_{\rho(u,v)}$

– points inward from $\mathbb{S}^2$.

(✱) ρ generates orientation opposite to given one.

$$\Rightarrow \iint_{\mathbb{S}^2} F\cdot d\vec{A} = -\iint_{(0,2\pi)\times(0,\pi)} \underbrace{F(\rho(u,v))\cdot[\partial_1\rho(u,v)\times\partial_2\rho(u,v)]_{\rho(u,v)}}\,du\,dv$$

$\rho(u,v)_{\rho(u,v)}\cdot(-\sin v)\,\rho(u,v)_{\rho(u,v)}$

$= -\sin v\,[\underbrace{\rho(u,v)\cdot\rho(u,v)}_{=1}] = -\sin v$

$$\Rightarrow \iint_{\mathbb{S}^2} F\cdot d\vec{A} = \int_0^{2\pi}du\int_0^{\pi}\sin v\,dv = 2\pi\cdot 2 = 4\pi.$$

Last part of module: applications of things we learned!

Problem: Given material for fence, total length 160m.

- you can fence off rectangular region, and keep enclosed land.
- What is the maximum area of land you can take?

- Assume: $2x^2+2y^2=160$ (Constraint)
- Maximise: x^2y^2 (optimisation)

More generally: constrained optimisation problem

(P1) Maximise/minimise $f(x,y)$, subject to constraint $g(x,y)=c$.

Q. What is the connection to geometry (and this module)?

(*) If g is nice enough, then

$$C=\{(x,y)\mid g(x,y)=c\} \text{ is a curve.}$$

$\Rightarrow$ (P1*) Maximise/minimise the function f on the curve C.

Setup: $U\subseteq\mathbb{R}^2$ – open, connected (usually $U=\mathbb{R}^2$)

(optimise this) • $f:U\to\mathbb{R}$ – smooth

(constraint) • $g:U\to\mathbb{R}$ – smooth

(constraint curve) • $C=\{(x,y)\in U\mid g(x,y)=c\}$

$\sim \nabla g(p)\neq\vec{0}_p$ for all $p\in C$ ($\Rightarrow C$ is a curve)

Main geometric observation is the following:

Thm: Assume the above setup. Suppose f achieves its $\overset{\text{maximum or minimum}}{\text{extremum}}$ value on C at $p\in C$. Then:

- $\nabla f(p)$ is normal to every element of T_pC.
- There exists $\lambda\in\mathbb{R}$ such that $\nabla f(p)=\lambda\nabla g(p)$.

Proof: Let $\gamma: I\to C$ be a parametrisation of C, with $\gamma(t_0)=p$.

$\Rightarrow f(\gamma(t))$ has max/min at $t=t_0$.

$$\Rightarrow 0=\frac{d}{dt}[f(\gamma(t))]\Big|_{t=t_0}=\ldots=\nabla f(p)\cdot\gamma'(t_0)_{\gamma(t_0)}$$

$\Rightarrow \nabla f(p) \cdot v_p = 0$ for all $v_p \in T_pC$ $(= s \cdot \gamma'(t_0)_{\gamma(t_0)})$

Note: $\nabla g(p)$ also normal to T_pC (and nonzero)

Since there is only one normal dimension to T_pC

$\Rightarrow \nabla f(p) = \lambda \nabla g(p)$ for some $\lambda \in \mathbb{R}$.

Q. Why does this help?

(A) To find extremum of f on C, we do not need to check all points (too many!)

— only points $p \in C$ satisfying $\nabla f(p) = \lambda \nabla g(p)$.

This leads to method of Lagrange multipliers:

- Step 1: solve system

$$\left.\begin{aligned} \partial_1 f(x,y) &= \lambda \partial_1 g(x,y) \\ \partial_2 f(x,y) &= \lambda \partial_2 g(x,y) \end{aligned}\right\} \nabla f(p) = \lambda \nabla g(p)$$

$$g(x,y) = c \quad \} \ (x,y) \in C$$

for unknowns (x, y, λ) ← Lagrange multiplier

- Step 2: For each solution (x,y,λ):

Check if f achieves maximum or minimum at (x,y)

"brute force"

will discuss further later

(A) (x,y) could correspond to $\begin{cases} \bullet \text{ max of } f \\ \bullet \text{ min of } f \\ \bullet \text{ neither} \end{cases}$