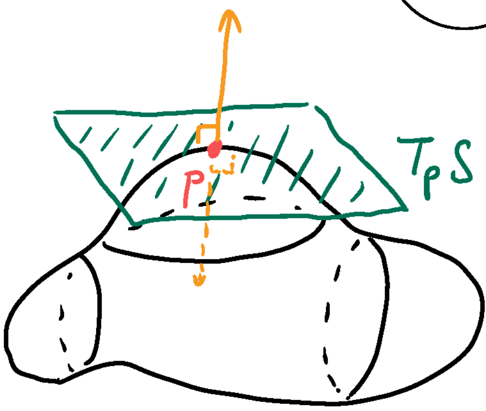
Last time: tangent planes to surfaces ($S \subseteq \mathbb{R}^n$)

Now: Special case $n=3$

$\Rightarrow$ Surface $S \subseteq \mathbb{R}^3$, and point $p \in S$.



- $\underbrace{T_pS}_{\text{tangent plane}}$: 2-d plane lying in 3-d space

(more formally, 2-d subspace of $T_p\mathbb{R}^3$)

(★) There is one remaining dimension (in $T_p\mathbb{R}^3$), $\underbrace{\text{normal}}_{\text{"perpendicular"}}$ to T_pS.

~ useful to capture unit tangent vectors in this dimension.

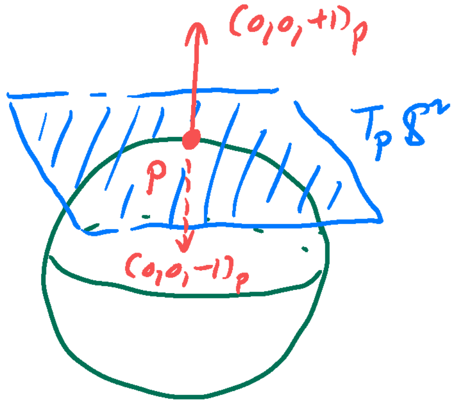
Def. Let $S \subseteq \mathbb{R}^3$ be a surface, and let $p \in S$. Then, $n_p \in T_p\mathbb{R}^3$ is a unit normal to S at p iff:

(1) $n_p \cdot v_p = 0$ for every $v_p \in T_pS$ (n_p normal to T_pS)

(2) $|n_p| = 1$ (n_p has unit length)

Ex. $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$p = (0,0,1)$ – north pole



- $T_p\mathbb{S}^2$ – "copy of xy-plane at $(0,0,1)$"
  - spanned by x and y directions.

$\Rightarrow T_p\mathbb{S}^2 = \{a \cdot (1,0,0)_{(0,0,1)} + b \cdot (0,1,0)_{(0,0,1)} \mid a,b \in \mathbb{R}\}$

$\Rightarrow$ Two directions normal to $T_p\mathbb{S}^2$:

- $(0,0,+1)_{(0,0,1)}$, $(0,0,-1)_{(0,0,1)}$ – unit normals to $\mathbb{S}^2$ at p

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Q. How can we compute unit normals?

(I) Can be computed using parametrisations.

Thm: Let S and p be as before, and let σ be a parametrisation of S, with $p = \sigma(u_0,v_0)$. Then, the unit normals to S at p are:

$$n_p^{\pm} = \pm \left[\frac{\partial_1\sigma(u_0,v_0) \times \partial_2\sigma(u_0,v_0)}{|\partial_1\sigma(u_0,v_0) \times \partial_2\sigma(u_0,v_0)|} \right]_{\sigma(u_0,v_0)}$$

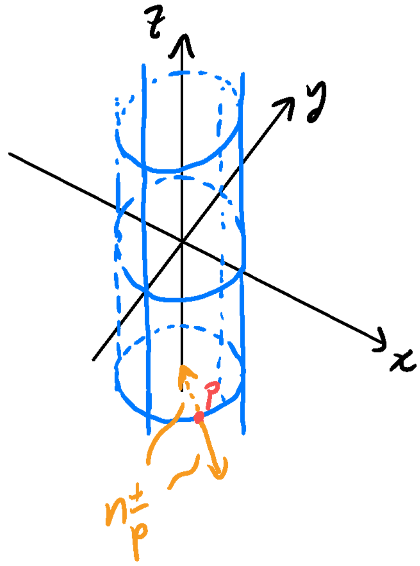
Proof: $\partial_1\sigma(u_0,v_0)_{\sigma(u_0,v_0)}, \partial_2\sigma(u_0,v_0)_{\sigma(u_0,v_0)}$ – basis for $T_{\sigma(u_0,v_0)} = T_pS$.

$\Longrightarrow [\partial_1\sigma(u_0,v_0) \times \partial_2\sigma(u_0,v_0)]_{\sigma(u_0,v_0)}$ – normal to T_pS.

$\sim$ divide by length, take $\pm$ $\Longrightarrow$ unit normals.

Ex. $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$

Find unit normals at $p = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1)$



Parametrise: $\sigma : \mathbb{R}^2 \to \mathcal{C}$, $\sigma(u,v) = (\cos u, \sin u, v)$

Note: $p = \sigma(-\frac{\pi}{4}, -1)$

Recall: $\partial_1\sigma(u,v) \times \partial_2\sigma(u,v) = (\cos u, \sin u, 0)$

$|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| = 1$

Thus, unit normals to p are:

$$n_p^{\pm} = \pm \left[\frac{(\cos u, \sin u, 0)}{1} \right]_{\underbrace{(\cos u, \sin u, v)}_{\sigma(u,v)}} \Bigg|_{(u,v) = (-\frac{\pi}{4}, -1)} = \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)_{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1)}$$

(II) For level sets, can be computed via gradients.

Thm: Assume the setting of the "level set theorem":

- $S = \{(x,y,z) \in V \mid f(x,y,z) = c\}$
- $V \subseteq \mathbb{R}^3$ – open, connected; $c \in \mathbb{R}$; $f : V \to \mathbb{R}$ – smooth
- $\nabla f(p) \neq \vec{0}_p$ for all $p \in S$.

(In particular, S is a surface.)

Then, at any $p \in S$, the unit normals to S at p are:

$$n_p^{\pm} = \pm \frac{1}{|\nabla f(p)|} \cdot \nabla f(p)$$

Proof: Let $\sigma : U \to S$ be a parametrisation of S, with $\sigma(u_0,v_0) = p$.

$\Rightarrow$ • $f(\sigma(u,v)) = c$ • $\partial_u[f(\sigma(u,v))] = 0$ • $\partial_v[f(\sigma(u,v))] = 0$

$\quad\quad\quad$ $\parallel$ (Chain rule) $\parallel$

$\nabla f(\sigma(u,v)) \cdot \partial_1\sigma(u,v)_{\sigma(u,v)}$ $\quad$ $\nabla f(\sigma(u,v)) \cdot \partial_2\sigma(u,v)_{\sigma(u,v)}$

Set $(u,v) = (u_0,v_0)$:

$$\Rightarrow \nabla f(p) \cdot [a \cdot \partial_1\sigma(u_0,v_0)_{\sigma(u_0,v_0)} + b \cdot \partial_2\sigma(u_0,v_0)_{\sigma(u_0,v_0)}] = 0 \quad (a,b \in \mathbb{R})$$

Thus, $\pm\nabla f(p)$ normal to every element of $T_{\sigma(u_0,v_0)} = T_pS$.

$\Rightarrow$ divide by $|\nabla f(p)|$ to finish proof.

Ex. $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2+y^2}_{f(x,y,z)} = 1\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1)$

~ $\mathcal{C}$ is a level set of f.

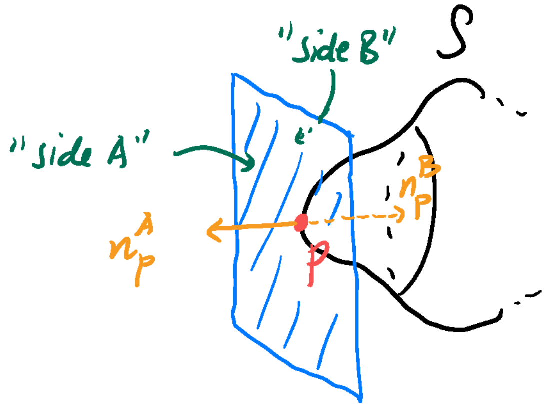
- $\nabla f(x,y,z) = (2x, 2y, 0)_{(x,y,z)}$
- $|\nabla f(x,y,z)| = \sqrt{4x^2+4y^2} = 2\underbrace{\sqrt{x^2+y^2}}_{=1 \text{ on } \mathcal{C}} = 2$ on $\mathcal{C}$.

$\Rightarrow$ unit normals at $(x,y,z) \in \mathcal{C}$:

$$n^{\pm}_{(x,y,z)} = \pm\frac{1}{2}(2x, 2y, 0)_{(x,y,z)} = (x,y,0)_{(x,y,z)}$$

- At p: $n^{\pm}_p = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)_{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1)}$ ~ Same as before!

Q. How to interpret unit normals?



Idea: A surface is "2-sided" at any $p \in S$.

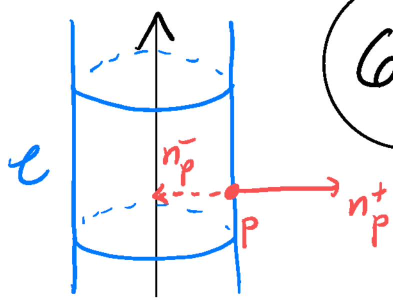
- T_pS is "2-sided"
- Each side represented by a unit normal to p.
- "Side of S at p" $\Leftrightarrow$ "Side of T_pS"

(*) Choice of unit normal at p $\Leftrightarrow$ "Choice of side of S at p"

Ex. $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$ – cylinder

Recall: At $p = (x,y,z) \in \mathcal{C}$, unit normals are $\pm(x,y,0)_{(x,y,z)}$.

- $n_p^+ = +(x,y,0)_{(x,y,z)}$ - points outward from $\mathcal{C}$
  $\Longrightarrow$ "outward-facing side of $\mathcal{C}$"
- $n_p^- = -(x,y,0)_{(x,y,z)}$ - points inward from $\mathcal{C}$
  $\Longrightarrow$ "inward-facing side of $\mathcal{C}$"



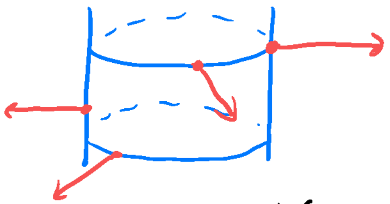
<u>Q.</u> Is S, as a whole, (globally) "2-sided"?

- "Side of all of S" $\Longleftrightarrow$ choice of unit normal n_p at <u>every</u> $p \in S$.

(∗) Choices must be consistent.
- n_p cannot suddenly jump to opposite side.

<u>Def.</u> Let $S \subseteq \mathbb{R}^3$ be a surface. An <u>orientation</u> of S is a choice of unit normal n_p to S at each $p \in S$, such that the n_p's vary continuously with respect to p.

<u>Ex.</u> $\mathcal{C}$ as before (cylinder)



- $n_{(x,y,z)} = +(x,y,0)_{(x,y,z)}$

"outward-facing orientation"

- $n_{(x,y,z)} = -(x,y,0)_{(x,y,z)}$

"inward-facing orientation"

2 sides of $\mathcal{C}$.

<u>Idea:</u> A parametrisation $\sigma: U \to S$ of S (locally) fixes a side of S.

- Choose $n_\sigma(u,v) = + \left[\dfrac{\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)}{|\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)|} \right]_{\sigma(u,v)}$ at each $(u,v) \in U$.

<u>Def.</u> Let $S \subseteq \mathbb{R}^3$ be a surface, and let O be an orientation of S.
- A parametrisation $\sigma: U \to S$ generates the orientation O iff $\boxed{n_\sigma(u,v)}$ coincides with O for all $(u,v) \in U$.
- σ generates an orientation opposite to O iff $n_\sigma(u,v)$ does not coincide with O for any $(u,v) \in U$.

Ex. $\mathcal{C}$ - cylinder

Parametrisation: $\sigma : \mathbb{R}^2 \to \mathcal{C}$, $\sigma(u,v) = (\cos u, \sin u, v)$

Recall: $n_\sigma(u,v) = + \left[\dfrac{\overbrace{\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)}^{(\cos u, \sin u, 0)}}{\underbrace{|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)|}_{1}} \right]_{\sigma(u,v)} = + (\cos u, \sin u, 0)_{(\cos u, \sin u, v)}$

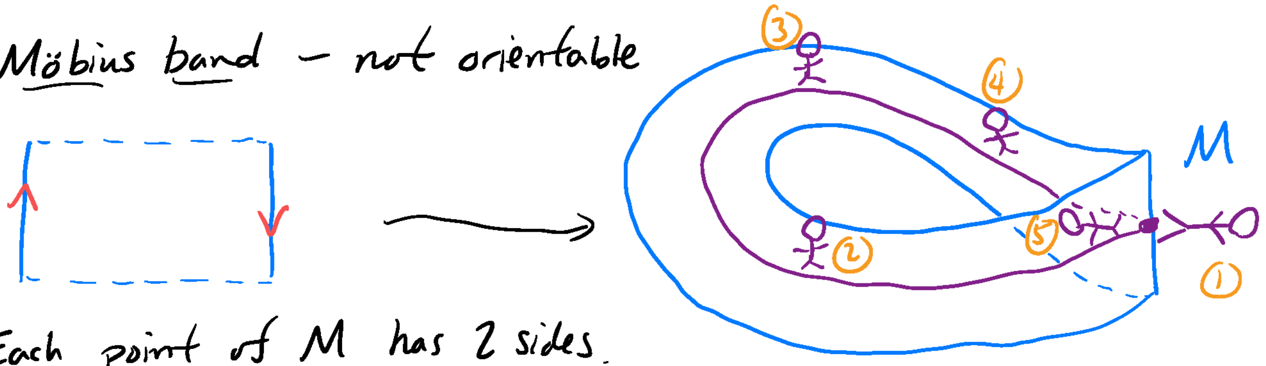
- At $(x,y,z) = \sigma(u,v)$ $\Rightarrow$ $n_\sigma(u,v) = + (x,y,0)_{(x,y,z)}$
  $(= (\cos u, \sin u, v))$

$\Rightarrow$ σ generates "outward-facing" orientation of $\mathcal{C}$

Def. A surface $S \subseteq \mathbb{R}^3$ is orientable
$\Leftrightarrow$ there exists an orientation of S.) – "S is globally 2-sided"

Ex. Cylinder $\mathcal{C}$, sphere $\mathbb{S}^2$ are orientable
~ Have outward-facing, inward-facing sides.

Ex. Möbius band – not orientable



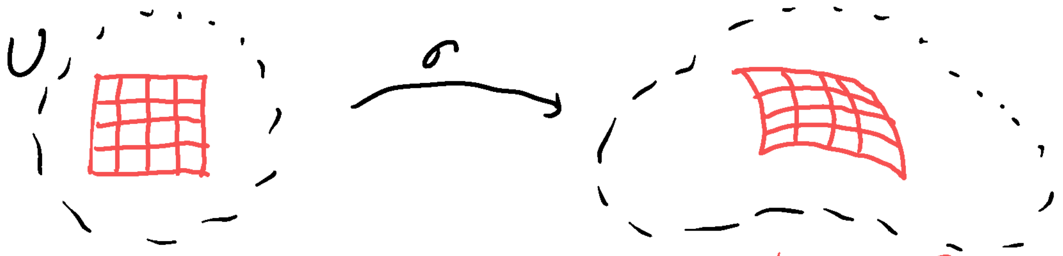
Each point of M has 2 sides.
- But, M as a whole does not!
- Cannot choose n_p continuously for all $p \in M$.

Q. How to define area ("size") of a surface?
- Also, "weighted areas" ~ integrals over surfaces.

(**Recall:** Arc length of curves – approximate using line segments.
- Infinitely good approximation $\Rightarrow$ exact length.)

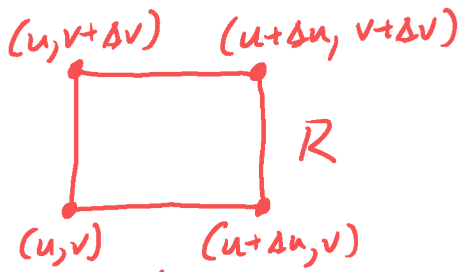
Want: Similar process for surface area.

First, consider parametric surface $\sigma : U \to \mathbb{R}^3$.

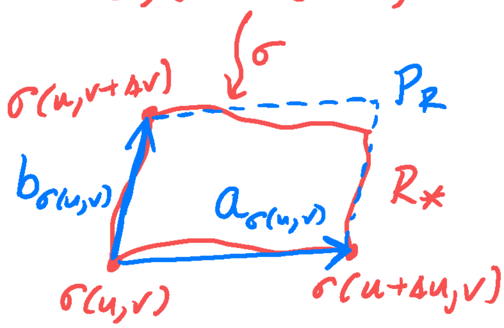


rectangles R in U "curved rectangles R_* in σ"

Then: $\text{Area}(\sigma) = \sum_{\text{rectangles } R} \text{Area}(R_*)$



Idea: Approximate $\text{Area}(R_*)$ by area of parallelogram P_R.

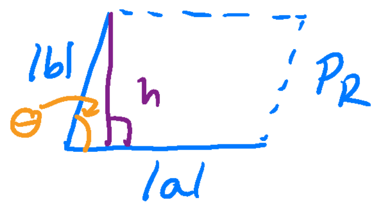


Sides of P_R given by:

- $a = \sigma(u+\Delta u, v) - \sigma(u,v)$
- $b = \sigma(u, v+\Delta v) - \sigma(u,v)$

Claim: $A(P_R) = |a \times b|$

(Proof: $A(P_R) = |a|\, h = |a||b| \sin\theta$)



Thus, $\text{Area}(\sigma) \approx \sum_{\text{rectangles } R} A(P_R)$

$$= \sum_{\text{rectangles } R} \left| \frac{a}{\Delta u} \times \frac{b}{\Delta v} \right| \Delta u \Delta v$$

"$\Delta u, \Delta v \searrow 0$"

$$\iint_U |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)| \, du\, dv$$

- To get exact value of $A(\sigma)$, let $\Delta u, \Delta v \searrow 0$

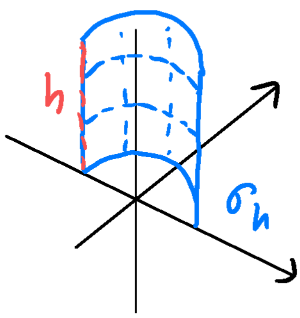
Def. Let $\sigma: U \to \mathbb{R}^3$ be a parametric surface. The surface area of σ is defined as

$$A(\sigma) = \iint_U |\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)| \, du\, dv.$$

Ex. Let $h > 0$, and $\sigma_h : (0,\pi) \times (0,h) \to \mathbb{R}^3$

$$\sigma_h(u,v) = (\cos u, \sin u, v)$$

(half-cylinder of height h)



- Find surface area of σ_h.

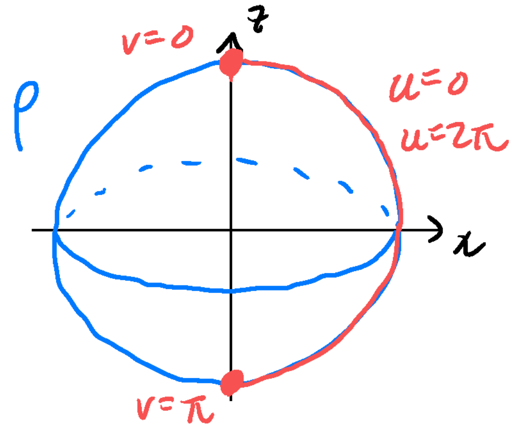
Recall: $|\partial_1\sigma_h(u,v)\times\partial_2\sigma_h(u,v)| = |(\cos u, \sin u, 0)| = 1$

$$\Rightarrow A(\sigma) = \iint_{(0,\pi)\times(0,h)} \underbrace{|\partial_1\sigma_h(u,v)\times\partial_2\sigma(u,v)|}_{1}\,du\,dv$$

$$= \int_0^\pi\int_0^h 1\,du\,dv = \underline{\pi h}$$

Ex. $\rho : (0,2\pi)\times(0,\pi)\to\mathbb{R}^3$, $\rho(u,v) = (\cos u\sin v, \sin u\sin v, \cos v)$

("almost all of the sphere")
(except red part)



Recall: $|\partial_1\rho(u,v)\times\partial_2\rho(u,v)| = |\sin v|$

- $0<v<\pi \Rightarrow \sin v > 0$

Thus: $$A(\rho) = \iint_{(0,2\pi)\times(0,\pi)} \sin v\,du\,dv$$

$$= \int_0^{2\pi}du\int_0^\pi \sin v\,dv = 2\pi\underbrace{\left[-\cos\pi\right]_{v=0}^{v=\pi}}_{2} = \underline{4\pi}.$$

(*) Thus, area of sphere (minus "a bit") is $4\pi = 4\pi\cdot 1^2$.

Next, consider "weighted area" over (parametric) surfaces.

- $\sigma : U\to\mathbb{R}^3$ – parametric surface.

- $$\iint_U \underbrace{\qquad}_{\uparrow}\ \underbrace{|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|}_{\text{area of "infinitesimal parallelogram" at } \sigma(u,v)}\,du\,dv$$

- Add weight F to integral
- Weight should be at $\sigma(u,v)$.

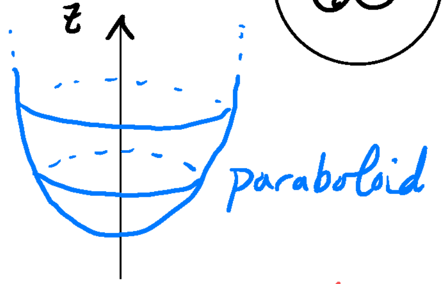
Def Let $\sigma : U\to\mathbb{R}^3$ be a parametric surface, and let F be a real-valued function defined on the image of σ. Then, the <u>surface integral</u> of F over σ is defined as

$$\iint_\sigma F\,dA = \iint_U F(\sigma(u,v))\,|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|\,du\,dv$$

Rmk. $A(\sigma) = \iint_\sigma 1 \cdot dA$

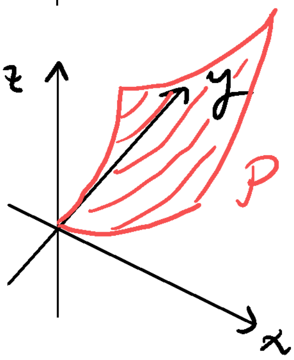
Ex. $P: (0,1)\times(0,1) \to \mathbb{R}^3$, $P(u,v) = (u, v, u^2+v^2)$

(part of paraboloid)



$H: \mathbb{R}^3 \to \mathbb{R}$, $H(x,y,z) = \sqrt{1+2x^2+2y^2+2|z|}$

- Find $\iint_P H\, dA$.



Direct computation:

- $|\partial_1 P(u,v) \times \partial_2 P(u,v)| = \sqrt{1+4u^2+4v^2}$
- $H(P(u,v)) = H(u,v,u^2+v^2) = \sqrt{1+4u^2+4v^2}$

$$\implies \iint_P H\, dA = \iint_{(0,1)\times(0,1)} \underbrace{H(P(u,v))}_{\sqrt{1+4u^2+4v^2}}\ \underbrace{|\partial_1 P(u,v) \times \partial_2 P(u,v)|}_{\sqrt{1+4u^2+4v^2}}\, du\, dv$$

$$= \int_0^1 \int_0^1 (1+4u^2+4v^2)\, du\, dv$$

$$= \int_0^1 \left(1 + \frac{4}{3} + 4v^2\right) dv$$

$$= 1 + \frac{4}{3} + \frac{4}{3} = \frac{11}{3}.$$