Last time: When does a parametric surface $\sigma: U \to \mathbb{R}^n$ fail to describe a 2-d "surface"?

Def. Let $\sigma: U \to \mathbb{R}^n$ be a parametric surface. Then, σ is regular iff $\partial_1\sigma(u,v)$, $\partial_2\sigma(u,v)$ are linearly independent for all $(u,v) \in U$.

Tangent plane $T_\sigma(u,v)$ is 2-d for all $(u,v) \in U$

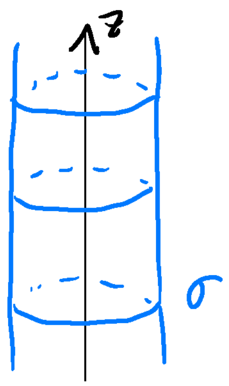
Q. Is there a computational way to check if σ is regular?

A. Yes, if $n=3$ (σ in 3-d space)

Thm: A parametric surface $\sigma: U \to \mathbb{R}^3$ is regular if and only if $|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| \neq 0$ for all $(u,v) \in U$.

(Proof: $\partial_1\sigma(u,v), \partial_2\sigma(u,v)$ linearly independent $\iff$ $\partial_1\sigma(u,v) \times \partial_2\sigma(u,v) \neq \vec{0}$.)

Ex. $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$ (cylinder)



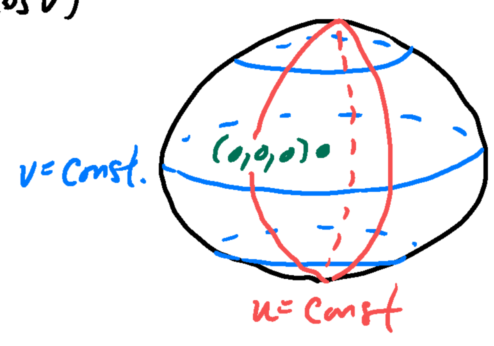
- $\partial_1\sigma(u,v) = (-\sin u, \cos u, 0)$ • $\partial_2\sigma(u,v) = (0,0,1)$
- $\partial_1\sigma(u,v) \times \partial_2\sigma(u,v) = (\cos u, \sin u, 0)$

$\Rightarrow |\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| = 1$ for all $(u,v) \in \mathbb{R}^2$

$\Rightarrow$ σ is regular

Ex: $\rho: \mathbb{R}^2 \to \mathbb{R}^3$, $\rho(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

(spherical coordinates)



$$|\rho(u,v)| = \sqrt{\underbrace{\cos^2 u \sin^2 v + \sin^2 u \sin^2 v}_{\sin^2 v} + \cos^2 v} = 1$$

$|\rho(u,v)|$: dist. 1 from origin

- $\partial_1\rho(u,v) = (-\sin u \sin v, \cos u \sin v, 0)$ • $\partial_2\rho(u,v) = (\cos u \cos v, \sin u \cos v, -\sin v)$

$\Rightarrow \partial_1 \rho(u,v) \times \partial_2 \rho(u,v) = -\sin v\,(\cos u \sin v, \sin u \sin v, \cos v)$
$= -\sin v \cdot \rho(u,v)$

$\Rightarrow |\partial_1 \rho(u,v) \times \partial_2 \rho(u,v)| = |\sin v|$ ($=0$ when $v = 0, \pm\pi, \pm 2\pi, \ldots$)
(at north/south poles)

Thus, ρ is not regular.

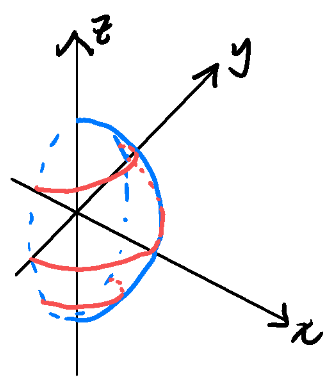
Remark: ρ can be made regular by restricting domain to avoid $v = 0, \pm\pi, \ldots$, e.g. $\rho : \mathbb{R}_u \times (0,\pi)_v \to \mathbb{R}^3$

Another issue: different parametric surfaces could describe same "2-d geometric object".
(Analogous to previous discussion for parametric curves.)

Def. Let $\sigma : U \to \mathbb{R}^n$ and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^n$ be regular parametric surfaces. We say that σ is a reparametrisation of $\tilde{\sigma}$ iff there exists a bijection $\Phi : U \longleftrightarrow \tilde{U}$ such that:
(1) Both Φ and Φ^{-1} are smooth.
(2) $\tilde{\sigma}(\Phi(u,v)) = \sigma(u,v)$ for all $(u,v) \in U$.
Also, Φ is called the corresponding change of variables.

Interpretation: $\sigma, \tilde{\sigma}$ map to the same points.
- $(\tilde{u}, \tilde{v}) = \Phi(u,v) \Longrightarrow \tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(u,v)$
- Φ - describes how (u,v) and $(\tilde{u}, \tilde{v})$ are related.

Ex. $\rho : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \pi) \to \mathbb{R}^3$ (half-sphere)
$\rho(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$



$\tilde{\rho} : \underbrace{B(\vec{0}, 1)}_{\{(u,v) \in \mathbb{R}^2 \,|\, u^2 + v^2 < 1\}} \to \mathbb{R}^3$, $\tilde{\rho}(\tilde{u}, \tilde{v}) = (\sqrt{1 - \tilde{u}^2 - \tilde{v}^2}, \tilde{u}, \tilde{v})$

- If we set $(\tilde{u}, \tilde{v}) = \Phi(u,v) = (\sin u \sin v, \cos v)$

$\Rightarrow \tilde{\rho}(\tilde{u}, \tilde{v}) = (\underbrace{\sqrt{1 - \sin^2 u \sin^2 v - \cos^2 v}}, \sin u \sin v, \cos v) = \rho(u,v)$

$\sqrt{\sin^2 v - \sin^2 u \sin^2 v} = \sqrt{\sin^2 v \cos^2 u} = \cos u \sin v) > 0$

(Also, can show Φ is bijection, and Φ, Φ^{-1} are smooth.)

$\Rightarrow$ ρ is a reparametrisation of $\tilde{\rho}$.

(equivalently, ρ and $\tilde{\rho}$ are reparametrisations of each other)

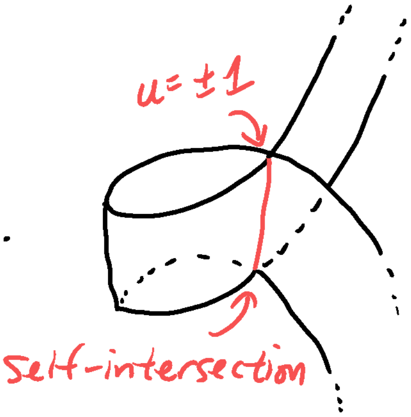
(*) $\rho, \tilde{\rho}$ are different parametric surfaces, but describe same half-sphere.

Finally: Want to exclude self-intersections.

Ex. $\sigma : \mathbb{R} \times (0,1) \to \mathbb{R}^3$

$\sigma(u,v) = (u^2 - 1, u^3 - u, v)$

Should not describe a surface.
(by choice)



Like for curves, we have main principles for defining "surfaces".

[1] Described using parametric surfaces.
[2] Independent of parametrisation.
[3] Not self-intersecting.

Def. $S \subseteq \mathbb{R}^n$ is a surface iff for any $p \in S$, there exist:

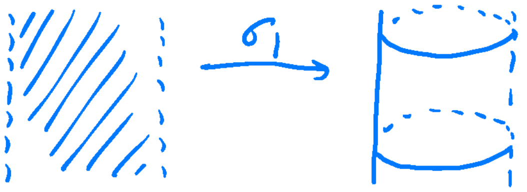
(i) An open subset $V \subseteq \mathbb{R}^n$ such that $p \in V$, and
(ii) A regular and injective parametric surface $\sigma : U \to S$,

such that the following hold:

(a) σ is a bijection between U and $S \cap V$, and
(b) The inverse $\sigma^{-1} : S \cap V \to U$ of σ is also continuous.

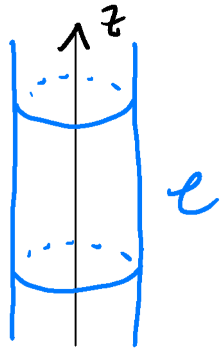
Informal idea: A surface is constructed by "gluing together" "deformed 2-d regions".

Ex. $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ (cylinder)



$\sigma_1 : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$
$\sigma_1(u,v) = (\cos u, \sin u, v)$

$\sigma_2 : (-\pi, \pi) \to \mathbb{R} \to \mathbb{R}^3$
$\sigma_2(u,v) = (\cos u, \sin u, v)$



<u>Idea</u>: σ_1, σ_2 "roll" flat strip into cylinder (except for one vertical line)

(*) Full cylinder $\mathcal{C}$ obtained by "gluing together" images of σ_1, σ_2.

<u>Next</u>: More practical ways to describe/construct surfaces.
(~ Analogous to discussion on curves.)

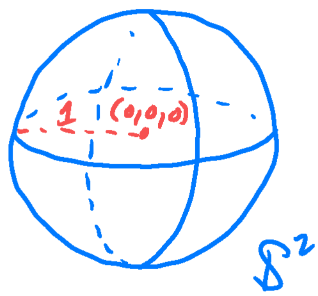
<u>Def.</u> Let $S \subseteq \mathbb{R}^n$ be a surface. Any regular parametric surface $\sigma : U \to S$ (not necessarily injective) is called a <u>parametrisation</u> of S.

<u>Ex.</u> $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ – cylinder
$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \; \sigma(u,v) = (\cos u, \sin u, v)$

- σ is a regular parametric surface.
- Each value $\sigma(u,v)$ is in $\mathcal{C}$.

$\Rightarrow \sigma$ is a parametrisation of $\mathcal{C}$
~ image of σ is all of $\mathcal{C}$, but σ is not injective.

<u>Ex.</u> $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ (sphere)
also is a surface



$\rho : \mathbb{R} \times (0, \pi) \to \mathbb{S}^2$
$\rho(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

- <u>Recall</u>: $|\partial_1 \rho(u,v) \times \partial_2 \rho(u,v)| = |\sin v| \underset{0<v<\pi}{\neq} 0$ $\Rightarrow$ ρ is regular

Thus: ρ is a parametrisation of $\mathbb{S}^2$.
(image of ρ is $\mathbb{S}^2 \setminus \{\underbrace{(0,0,\pm 1)}_{\text{2 points} \sim \text{north/south poles.}}\}$)

(*) In fact, there is no (regular) parametrisation whose image is all of $\mathbb{S}^2$!

Q. How can one show something is a surface?

(I) Nice level sets in $\mathbb{R}^3$ are surfaces.

Thm. Let $V \subseteq \mathbb{R}^3$ be open and connected, and let $f : V \to \mathbb{R}$ be smooth. Also, let $c \in \mathbb{R}$, and let
$$S = \{(x,y,z) \in V \mid f(x,y,z) = c\}$$ – level set of f

If $\nabla f(p)$ is nonzero for every $p \in S$, then S is a surface.

(Proof again uses implicit function theorem.)

Ex. Let $s : \mathbb{R}^3 \to \mathbb{R}$, $\quad s(x,y,z) = x^2 + y^2 + z^2$.

(*) Sphere $\mathbb{S}^2$ is a level set of s:
$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid s(x,y,z) = 1\}$$

- $\nabla s(x,y,z) = (2x, 2y, 2z)_{(x,y,z)}$ ($= (0,0,0)_{(x,y,z)}$ only when $(x,y,z) = (0,0,0)$)
- Since $(0,0,0) \notin \mathbb{S}^2$ ($0^2 + 0^2 + 0^2 \neq 1$), then $\nabla s(p) \neq (0,0,0)_p$ for all $p \in \mathbb{S}^2$.

$\Rightarrow$ By "level set theorem", $\mathbb{S}^2$ is a surface.

(II) Graphs are surfaces

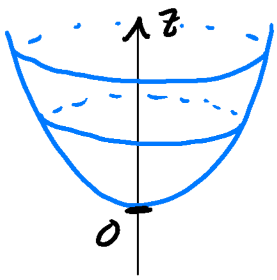
Cor. Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $f : U \to \mathbb{R}$ be smooth. Then, $G_f = \{(u, v, f(u,v)) \mid (u,v) \in U\}$ (the graph of f) is a surface.

Proof: Idea: $G_f = \{(x,y,z) \in U \times \mathbb{R} \mid \underbrace{z - f(x,y)}_{\text{has nonzero gradient}} = 0\}$ – "nice level set"

Remark: $\sigma_f : U \to G_f$, $\sigma_f(u,v) = (u,v,f(u,v))$ is an injective parametrisation of G_f, whose image is all of G_f.

Ex. $q : \mathbb{R}^2 \to \mathbb{R}$, $q(x,y) = x^2+y^2$

$G_q = \{ (u,v,\underbrace{u^2+v^2}_{q(u,v)}) \mid (u,v) \in \mathbb{R}^2 \}$ is a surface (paraboloid)



Note: So far, all examples of surfaces lie in $\mathbb{R}^3$

Q. Are there "novel" surfaces in $\mathbb{R}^n$, $n>3$, that cannot be constructed in $\mathbb{R}^3$? (**A.** Yes!)

Bonus Ex. (Klein Bottle)

Can be constructed by gluing edges of a rectangle.

(*) Cannot be realised in $\mathbb{R}^3$ without self-intersection

- Can be done in $\mathbb{R}^4$!

Q. Are there surfaces in $\mathbb{R}^5$ that cannot be made in $\mathbb{R}^4$? (**A.** No!)

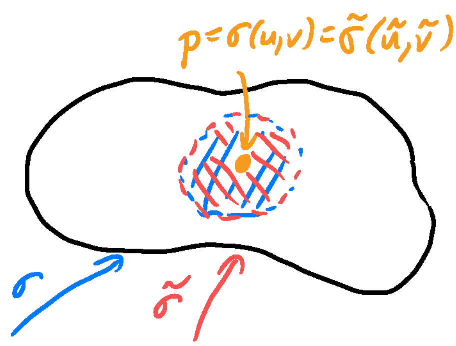
(*) Whitney embedding theorem: Every "type" of surface can be realised as a subset of $\mathbb{R}^4$.

Geometric Properties of Surfaces

- Measured using parametrisations.
- Must be independent of parametrisations.

Consider: tangent planes

- $S \subseteq \mathbb{R}^n$ - surface
- $\sigma : U \to S$, $\tilde{\sigma} : \tilde{U} \to S$ - parametrisations of S (2 ways to map out points of S)



- $T_\sigma(u,v)$ – "velocities along σ at p"
  // same!
- $T_{\tilde\sigma}(\tilde u,\tilde v)$ – "velocities along $\tilde\sigma$ at p"

$\Rightarrow$ Thus, expect $T_\sigma(u,v) = T_{\tilde\sigma}(\tilde u,\tilde v)$.
~ tangent planes should be independent of parametrisation.

<u>Thm</u>. Let $\sigma: U \to \mathbb{R}^n$ and $\tilde\sigma: \tilde U \to \mathbb{R}^n$ be regular parametric surfaces. Assume σ is a reparametrisation of $\tilde\sigma$, with change of variables $\Phi: U \leftrightarrow \tilde U$. $\left(\tilde\sigma(\Phi(u,v)) = \sigma(u,v) \right)$

Then, for any $(u,v) \in U$, $\quad T_\sigma(u,v) = T_{\tilde\sigma}(\Phi(u,v))$.

Proof: Write $\Phi(u,v) = (\tilde u(u,v), \tilde v(u,v))$

$\Rightarrow \sigma(u,v) = \tilde\sigma(\tilde u(u,v), \tilde v(u,v))$

Chain rule: $\bullet\ \partial_1\sigma(u,v) = \frac{\partial}{\partial u}[\tilde\sigma(\tilde u(u,v),\tilde v(u,v))]$

$$= \partial_1\tilde\sigma(\Phi(u,v))\cdot\partial_1\tilde u(u,v) + \partial_2\tilde\sigma(\Phi(u,v))\cdot\partial_1\tilde v(u,v)$$

– linear combination of $\partial_1\tilde\sigma(\tilde u,\tilde v)$, $\partial_2\tilde\sigma(\tilde u,\tilde v)$

$\bullet\ \partial_2\sigma(u,v) = \partial_1\tilde\sigma(\Phi(u,v))\cdot\partial_2\tilde u(u,v) + \partial_2\tilde\sigma(\Phi(u,v))\cdot\partial_2\tilde v(u,v)$

Switch σ and $\tilde\sigma$, replace Φ with Φ^{-1}:

$\Rightarrow \partial_1\tilde\sigma(\tilde u,\tilde v), \partial_2\tilde\sigma(\tilde u,\tilde v)$ – linear combinations of $\partial_1\sigma(u,v), \partial_2\sigma(u,v)$

Thus, $\underbrace{\mathrm{span}\{\partial_i\sigma(u,v)_{\sigma(u,v)} \mid i=1,2\}}_{T_\sigma(u,v)} = \underbrace{\mathrm{span}\{\partial_i\tilde\sigma(\Phi(u,v))_{\tilde\sigma(\Phi(u,v))}\}}_{T_{\tilde\sigma}(\Phi(u,v))}$

same point

(*) Thus, can define tangent planes to surfaces
(not just parametric surfaces)

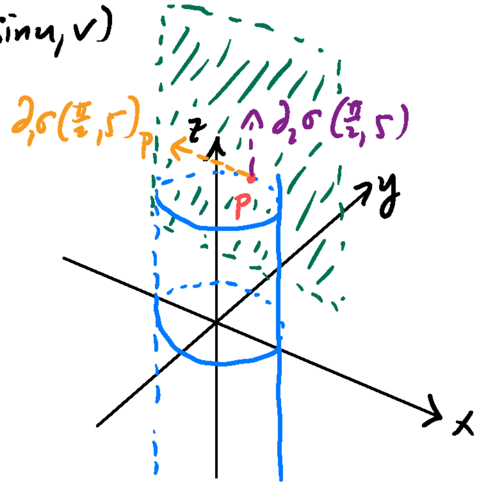
<u>Def.</u> Let $S \subseteq \mathbb{R}^n$ be a surface, and let $p \in S$. The <u>tangent plane</u> to S at p is defined as $T_pS = T_\sigma(u_0,v_0)$, where σ is <u>any</u> parametrisation of S, with $\sigma(u_0,v_0) = p$.

<u>Ex.</u> $\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$ – cylinder
Find tangent plane to $\mathcal{C}$ at $p = (0,1,5)$.

<u>Step 1</u>: Parametrise $\mathcal{C}$.

Recall: $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$

- $(0,1,5) = \sigma(\frac{\pi}{2}, 5)$



Step 2: Compute:

- $\partial_1\sigma(\frac{\pi}{2},5) = (-1,0,0)$
- $\partial_2\sigma(\frac{\pi}{2},5) = (0,0,1)$

$$\Rightarrow T_{(0,1,5)}\mathcal{C} = T_\sigma(\tfrac{\pi}{2},5)$$

$$= \{a\cdot(-1,0,0)_{(0,1,5)} + b\cdot(0,0,1)_{(0,1,5)} \mid a,b\in\mathbb{R}\}$$

Connection to linear algebra:

Consider surface S, and $p \in S$.

- Given parametrisation σ of S, with $p = \sigma(u,v)$

$$\Rightarrow \{\partial_1\sigma(u,v)_p,\ \partial_2\sigma(u,v)_p\} \text{ - basis for } T_pS$$

- Given another parametrisation $\tilde{\sigma}$ of S, with $p = \tilde{\sigma}(\tilde{u},\tilde{v})$

$$\Rightarrow \{\partial_1\tilde{\sigma}(\tilde{u},\tilde{v})_p,\ \partial_2\tilde{\sigma}(\tilde{u},\tilde{v})_p\} \text{ - another basis for } T_pS.$$

(✱) Thus, change of parametrisation of S) - different view of S.

$\Rightarrow$ change of basis of T_pS) - different view of T_pS.