Recall definition of curve integrals.

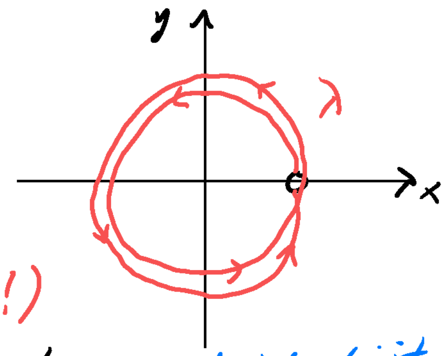
Q. What about the extra assumptions?

- Injective parametrisation
- Image differs by only finite number of points.

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ – circle.

$\lambda : (0, 4\pi) \to \mathcal{C}$, $\lambda(t) = (\cos t, \sin t)$

– parametrisation of all of $\mathcal{C}$.



Then: $L(\lambda) = \int_0^{4\pi} \underbrace{|\lambda'(t)|}_{1} dt = \underline{4\pi}$.

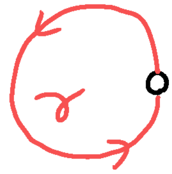
(omg!! not 2π!)

- λ hits (almost) every point of $\mathcal{C}$ twice! → need injectivity to avoid this!

(each point counted twice to length).

Consider instead $\gamma : (0, 2\pi) \to \mathcal{C}$, $\gamma(t) = (\cos t, \sin t)$.

– Injective, maps onto $\mathcal{C} \setminus \{(1,0)\}$.



$\Rightarrow L(\gamma) = \int_0^{2\pi} |\gamma'(t)|\, dt = \underline{2\pi}$. → as expected!

(A) γ missing the point $(1,0)$ is not an issue.

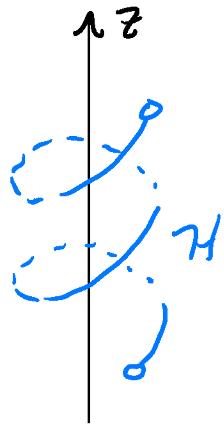
– A single point has zero length (won't make rigorous here).

$\Rightarrow$ Can allow image of γ to differ from $\mathcal{C}$ by single points.

Ex. $\mathcal{H} = \{(\cos t, \sin t, t) \mid t \in (-2\pi, 2\pi)\}$

$G : \mathbb{R}^3 \to \mathbb{R}$, $G(x,y,z) = x^2+y^2+z^2$

Find $\int_{\mathcal{H}} G\, ds$.



Injective parametrisation of all of $\mathcal{H}$.

- $h : (-2\pi, 2\pi) \to \mathcal{H}$, $h(t) = (\cos t, \sin t, t)$

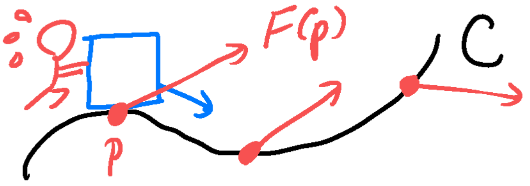
Then: $\int_{\mathcal{H}} G\, ds = \int_h G\, ds = \int_{-2\pi}^{2\pi} \underbrace{G(\cos t, \sin t, t)}_{1+t^2} \underbrace{|h'(t)|}_{\sqrt{2}}\, dt$

Previous example $\Rightarrow$ $= \sqrt{2}\left[4\pi + \frac{2}{3}(2\pi)^3\right]$.

Next: Curve integrals of vector fields

- direction also matters (rather than scalars)

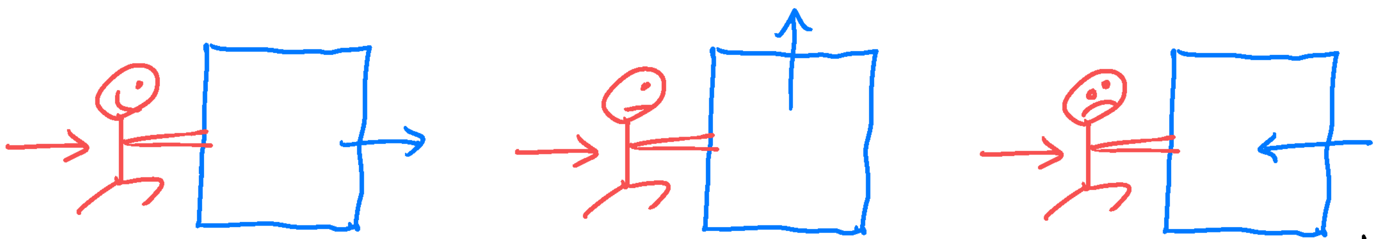
Physics motivation: "work" – "force applied over distance"



C – curve: path of box

F – vector field: force exerted on box

Q How "productive" was this force?



(positive work) (zero work) (negative work)

(A) only the component of F in the direction of C matters.
~ dot product of F with C-direction

– Which direction of C? (there are two!)
⇒ must choose orientation (i.e. unit tangents)

Def Let $C \subseteq \mathbb{R}^n$ be an oriented curve, and let F be a vector field on C. Then, we define the curve integral of F over C by:

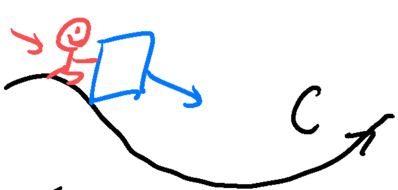
$$\int_C F \cdot d\vec{s} = \int_C \underbrace{(F \cdot T)}_{\text{scalar}}\, ds$$

curve integral of scalar function!

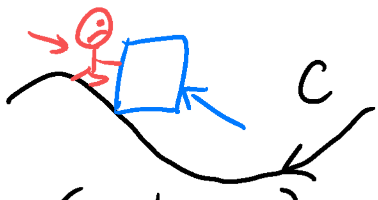
where T denotes the unit tangents of C in the direction specified by the orientation of C.

Q. Are these integrals geometric properties?

- Return to work example:



(work > 0)



(work < 0)

(※) Direction / orientation of C matters!

- If we reverse orientation of C.
  - T replaced by $-T$
  - Everything else independent of parametrisation

<u>Thm</u>: Let C and F be as before. Let C_* be the same curve as C, but with opposite orientation. Then:

$$\int_{C_*} F \cdot d\vec{s} = -\int_C F \cdot d\vec{s}.$$

(※) Curve integrals of vector fields are properties of <u>oriented curves</u>, not curves.

<u>Q</u>: How to compute these integrals? (via parametrisations)

<u>Thm</u>: Let C and F be as before, and let $\gamma : (a,b) \to C$ be an injective parametrisation of C, whose image differs from C by a finite number of points.

(1) If γ generates the orientation of C, then

$$\int_C F \cdot d\vec{s} = +\int_a^b [F(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt$$

(2) If γ generates the orientation opposite to that of C, then

$$\int_C F \cdot d\vec{s} = -\int_a^b [F(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt$$

Proof: (1) In the case, unit tangent along C is

$$T = +\frac{1}{|\gamma'(t)|} \cdot \gamma'(t)_{\gamma(t)}.$$

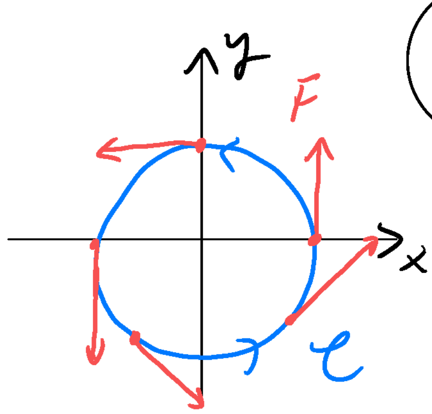
$$\Rightarrow \int_C F \cdot d\vec{s} = \int_C (F \cdot T)$$

$$= \int_a^b \left[F(\gamma(t)) \cdot \frac{1}{\cancel{|\gamma'(t)|}} \gamma'(t)_{\gamma(t)}\right] \cancel{|\gamma'(t)|}\, dt$$

(Proof of (2) is similar – replace "+" by "–".)

(※) Must be aware of <u>orientation</u> of parametrisations.

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ - circle
- with anticlockwise orientation



F - vector field on $\mathbb{R}^2$

$F(x,y) = (-y,x)_{(x,y)}$

Compute $\int_{\mathcal{C}} F \cdot d\vec{s}$.

Appropriate parametrisation: $\gamma : (0,2\pi) \to \mathcal{C}$, $\gamma(t) = (\cos t, \sin t)$
- Injective, covers all of $\mathcal{C}$ except one point.
- Generates anticlockwise orientation.

Observe: ~ $F(\gamma(t)) = (-\sin t, \cos t)_{(\cos t, \sin t)}$

~ $\gamma'(t)_{\gamma(t)} = (-\sin t, \cos t)_{(\cos t, \sin t)}$

~ $F(\gamma(t)) \cdot \gamma'(t) = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = 1$

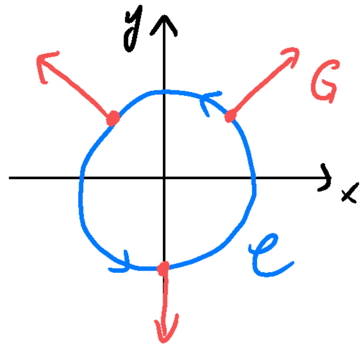
Thus, $$\int_{\mathcal{C}} F \cdot d\vec{s} = + \int_0^{2\pi} [F(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt$$

$$= \int_0^{2\pi} 1\, dt = \underline{2\pi}.$$

(Note: Direction of F same as direction along $\mathcal{C}$.)

Ex. $\mathcal{C}$ - as before, G - vector field on $\mathbb{R}^2$

$G(x,y) = (x,y)_{(x,y)}$

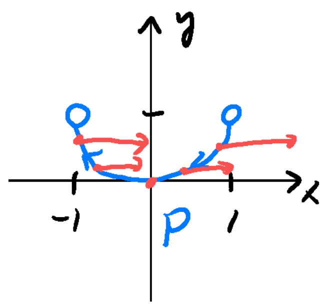


- Can use γ again
- $G(\gamma(t)) = (\cos t, \sin t)_{(\cos t, \sin t)}$

$$\Rightarrow \int_{\mathcal{C}} G \cdot d\vec{s} = + \int_0^{2\pi} \underbrace{[G(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]}_{(\cos t, \sin t) \cdot (-\sin t, \cos t) = 0}\, dt = \underline{0}$$

(Direction of G normal to direction along $\mathcal{C}$.)

Ex. $P = \{(t,t^2) \in \mathbb{R}^2 \mid -1 < t < 1\}$ (part of parabola)
- oriented in direction of decreasing x-value.



F - vector field on $\mathbb{R}^2$, $F(x,y) = (y,0)_{(x,y)}$.

Find: $\int_P F \cdot d\vec{s}$.

First step: parametrise P

- $\gamma : (-1,1) \to P$, $\gamma(t) = (t, t^2)$
  - injective parametrisation of all of P.
  - generates wrong (increasing x) orientation.

Compute: $\cdot\, F(\gamma(t)) = (t^2, 0)_{(t,t^2)}$ $\quad \cdot\, \gamma'(t)_{\gamma(t)} = (1, 2t)_{(t,t^2)}$

Thus: $\displaystyle\int_P F \cdot d\vec{s} = -\int_{-1}^{1} \underbrace{[F(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]}_{(t^2,0)\cdot(1,2t) = t^2} \, dt$

$$= -\int_{-1}^{1} t^2 \, dt = -\frac{2}{3}$$

Remark In other texts, the integral $\displaystyle\int_C F \cdot ds$ $\left(F(p) = (F_1(p), F_2(p))_p\right)$ is often written as $\displaystyle\int_C (F_1 dx + F_2 dy)$ (similar in higher dimensions)

Surfaces – "2-dimensional geometric objects"

(e.g. "surface of the earth", "piece of paper")

Plan: (1) Precise definition of surfaces.
(2) Geometric properties of surfaces.
(3) Surface integration.

Similar to study of curves, but with some novel challenges!

Q. What exactly is a surface?

(• curves: started with parametric curves
• here, a similar start for surfaces
(but functions of 2 variables)

Def. A parametric surface is a smooth vector-valued function $\sigma : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^2$ is open and connected.

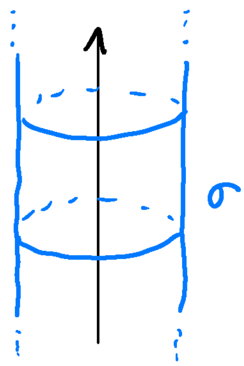
all partial derivatives exist

(Remark: For almost all our examples, $n = 3$.)

Ex. $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$

open, connected (cylinder) smooth

$\Rightarrow \sigma$ is a parametric surface.



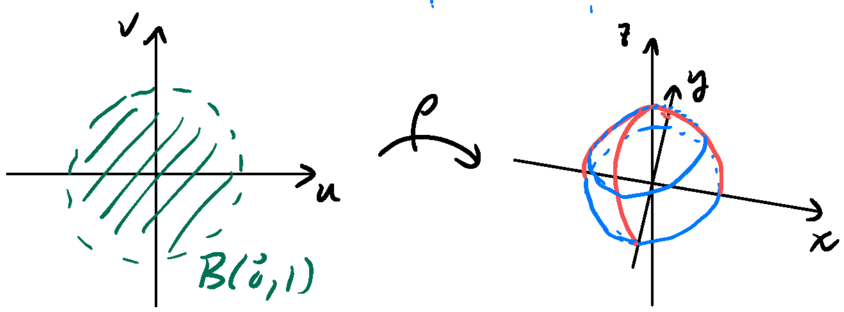
Ex. $\rho: B(\vec{0},1) \to \mathbb{R}^3$, $\rho(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$\{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$

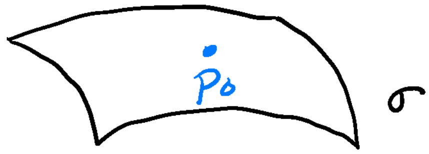
Image of ρ =

$\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1, z > 0\}$

– upper hemisphere

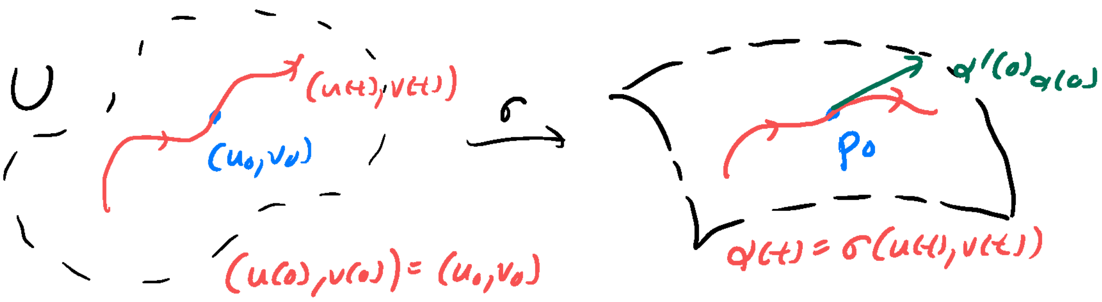


Consider a parametric surface $\sigma: U \to \mathbb{R}^n$, with $p_0 = \sigma(u_0, v_0)$



Suppose you are standing on p_0.

Q. With what velocities can you go along image of σ, starting from p_0? ("tangent directions" to σ at p_0)



– (∗) α – arbitrary path on σ through p_0

$\Rightarrow$ want: values $\alpha'(0)_{\alpha(0)}$ for all possible α.

- $\alpha'(0) = \frac{d}{dt}[\sigma(u(t),v(t))]\big|_{t=0}$

(chain rule) $= \partial_1\sigma(\underbrace{u(0),v(0)}_{(u_0,v_0)}) \cdot u'(0) + \partial_2\sigma(\underbrace{u(0),v(0)}_{(u_0,v_0)}) \cdot v'(0)$

$\Rightarrow \alpha'(0)_{\alpha(0)} = u'(0)\, \partial_1\sigma(u_0,v_0)_{p_0} + v'(0)\, \partial_2\sigma(u_0,v_0)_{p_0}$

~ all possible $\alpha'(0)_{\alpha(0)}$ = all linear combinations of $\partial_1\sigma(u_0,v_0)_{p_0}, \partial_2\sigma(u_0,v_0)_{p_0}$.

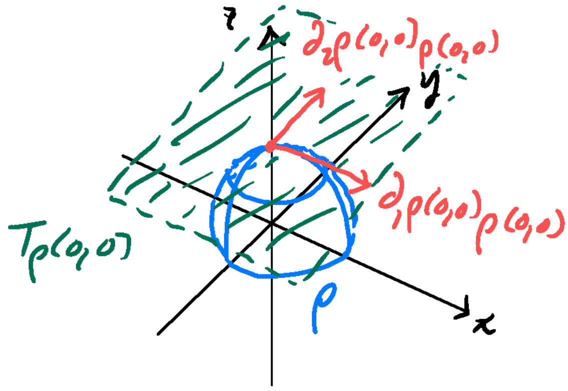
<u>Def.</u> Let $\sigma : U \to \mathbb{R}^n$ be a parametric surface, and let $(u_0,v_0) \in U$. We define the <u>tangent plane</u> to σ at (u_0,v_0) by:

$$T\sigma(u_0,v_0) = \{ a \cdot \partial_1\sigma(u_0,v_0)_{\sigma(u_0,v_0)} + b \cdot \partial_2\sigma(u_0,v_0)_{\sigma(u_0,v_0)} \mid a,b \in \mathbb{R} \}$$

<u>Remark:</u> $\underset{\displaystyle \text{span}\{\partial_1\sigma(u_0,v_0)_{\sigma(u_0,v_0)},\ \partial_2\sigma(u_0,v_0)_{\sigma(u_0,v_0)}\}}{\underset{\shortparallel}{\underline{T\sigma(u_0,v_0)}}}$ is a vector space (subspace of $T_{\sigma(u_0,v_0)}\mathbb{R}^n$).

<u>Ex.</u> $\rho : \underset{\displaystyle \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2<1\}}{\underset{\shortparallel}{\underline{B(\vec{0},1)}}} \to \mathbb{R}^3$, $\quad \rho(u,v) = (u, v, \sqrt{1-u^2-v^2})$

<u>Find:</u> $T\rho(0,0)$

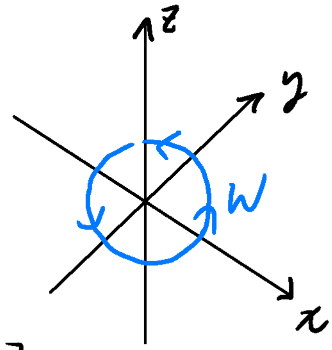


- $\partial_1\rho(u,v) = (1, 0, \tfrac{1}{2}(1-u^2-v^2)^{-1/2}(-2u))$
  $= \left(1, 0, \dfrac{-u}{\sqrt{1-u^2-v^2}}\right)$
- $\partial_2\rho(u,v) = \left(1, 0, \dfrac{-v}{\sqrt{1-u^2-v^2}}\right)$
- $\rho(0,0) = (0,0,1)$ $\quad$ • $\partial_1\rho(0,0) = (1,0,0)$ $\quad$ • $\partial_2\rho(0,0) = (0,1,0)$

$$\Rightarrow T\rho(0,0) = \underline{\{ a \cdot (1,0,0)_{(0,0,1)} + b \cdot (0,1,0)_{(0,0,1)} \mid a,b \in \mathbb{R} \}}$$

<u>Q.</u> How can parametric surfaces fail to describe surfaces?

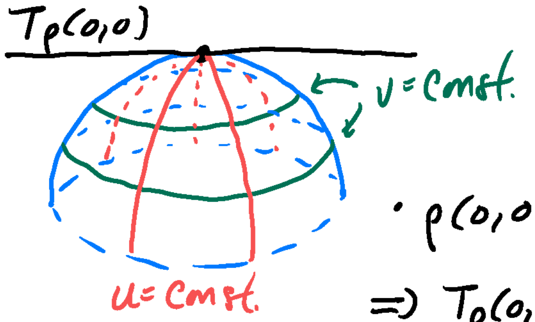
<u>Ex.</u> $\left.\begin{array}{l} W : \mathbb{R}^2 \to \mathbb{R}^3 \\ W(u,v) = (\cos u, \sin u, 0) \end{array}\right\}$ – image is unit circle in xy-plane



- $\partial_1 W(u,v) = (-\sin u, \cos u, 0)$
- $\partial_2 W(u,v) = (0,0,0)$
- $T_W(u,v) = \{ a \cdot (-\sin u, \cos u, 0)_{(\cos u, \sin u, 0)} \mid a \in \mathbb{R} \}$ – 1-d vector space

<u>Ex.</u> $\rho : \mathbb{R}^2 \to \mathbb{R}^3$, $\quad \rho(u,v) = \underbrace{(\cos u \sin v, \sin u \sin v, \cos v)}_{\text{Spherical Coordinates}}$

(image is $\underset{\text{2-d!}}{\underline{\text{the unit sphere}}}$!)



- $\partial_1\rho(u,v) = (-\sin u \sin v, \cos u \sin v, 0)$
- $\partial_2\rho(u,v) = (\cos u \cos v, \sin u \cos v, -\sin v)$

- $\rho(0,0) = (0,0,1)$ • $\partial_1\rho(0,0) = (0,0,0)$ • $\partial_2\rho(0,0) = (1,0,0)$

$\Rightarrow T_\rho(0,0) = \{ b\cdot(1,0,0)_{(0,0,1)} \mid b\in\mathbb{R} \}$) 1-d!

(*) Image of ρ is 2-d, but tangent plane only 1-d.
$\sim$ ρ is "defective" at $(u,v)=(0,0)$. → ($v=0$ is a point, not a circle

Idea: ρ "not defective" $\Leftrightarrow$ tangent planes are 2-d.

Def. Let $\sigma: U \to \mathbb{R}^n$ be a parametric surface. Then, σ is regular iff $\partial_1\sigma(u,v)$, $\partial_2\sigma(u,v)$ are linearly independent for all $(u,v)\in U$.

($T_\sigma(u,v)$ is 2-d for all $(u,v)\in U$

$\partial_1\sigma$, $\partial_2\sigma$ point in different directions.