Last time: properties of curve C at point $p \in C$.

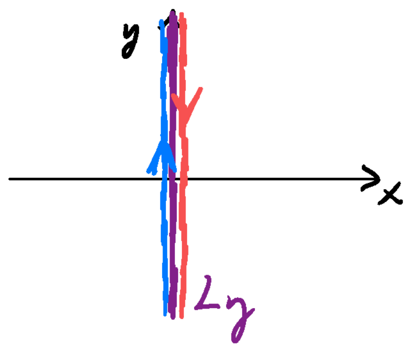
- tangent line (all possible velocities at p)
- unit tangents (directions at p)

Next: orientation – "direction along C"

Ex. $L_y = \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$

Can traverse L_y in two directions:

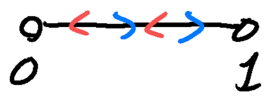
- up (increasing y)
- down (decreasing y)



Q. Why do we care? (Where is this used?)

Ex. (1) $\int_0^1 dx = +1$, $\int_1^0 dx = -1$

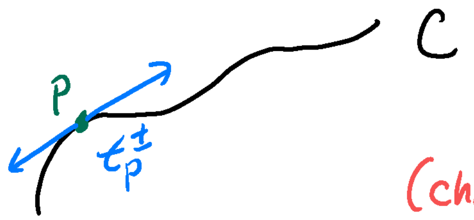
– depends on orientation of $(0,1)$



(2) Polar angles – which angles are positive/negative?

– Choice of orientation of circle.

Q. How to make this precise?



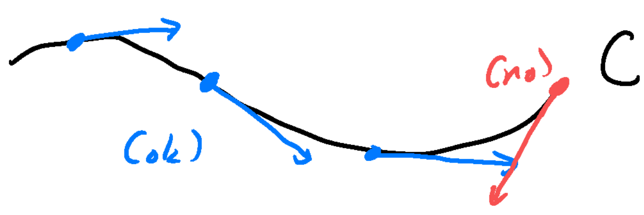
Recall: unit tangents $t_p^\pm$ to C at p

– Represent directions along C at p

(choice of unit tangent = choice of direction)

$\Rightarrow$ Direction along all of $C \iff$ Choice of unit tangent t_p at every $p \in C$

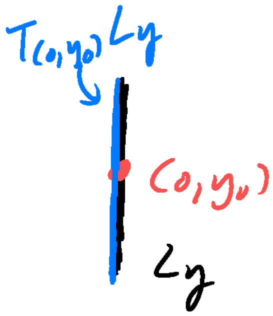
- Choices must be consistent (Cannot suddenly "jump" to opposite direction)



Def An orientation of a curve $C \subseteq \mathbb{R}^n$ is a choice of a unit tangent $t_p \in T_pC$ at each $p \in C$, such that the t_p's vary continuously with respect to p.

<u>Ex.</u> $L_y = \{(x,y)\in\mathbb{R}^2 \mid x=0\}$ – y-axis

Consider $(0,y_0)\in L_y$



- $T_{(0,y_0)}L_y = \{s\cdot(0,1)_{(0,y_0)} \mid s\in\mathbb{R}\}$

  $\Rightarrow$ unit tangents : $\pm(0,1)_{(0,y_0)}$

<u>Orientations</u>: (1) increasing y: choose $(0,+1)_{(0,y_0)}$ at each $(0,y_0)$

(2) decreasing y: choose $(0,-1)_{(0,y_0)}$ at each $(0,y_0)$

Orientations also captured through parametrisations:

– Parametrisations give direction of travel.

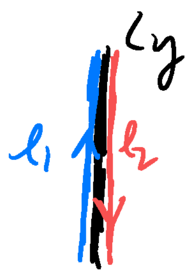
- Given parametrisation $\gamma: I\to C$ of C:
  - unit tangents given by $\pm\frac{1}{|\gamma'(t)|}\cdot\gamma'(t)_{\gamma(t)}$
  - "+" - direction along γ.

<u>Def.</u> Let $C\subseteq\mathbb{R}^n$ be a curve, and $\mathcal{O}$ an orientation of C.

- A parametrisation $\gamma: I\to C$ of C generates the orientation $\mathcal{O}$ iff $\frac{1}{|\gamma'(t)|}\cdot\gamma'(t)_{\gamma(t)}$ coincides with $\mathcal{O}$ for all $t\in I$.
- $\gamma: I\to C$ generates the orientation opposite to $\mathcal{O}$ iff $\frac{1}{|\gamma'(t)|}\cdot\gamma'(t)_{\gamma(t)}$ does not coincide with $\mathcal{O}$ for $t\in I$.

<u>Ex.</u> L_y – y-axis

$$\left.\begin{aligned} &\bullet\ \ell_1:\mathbb{R}\to L_y,\ \ell_1(t)=(0,t) \\ &\bullet\ \ell_2:\mathbb{R}\to L_y,\ \ell_2(t)=(0,-t)\end{aligned}\right\}\ \text{parametrisations of } L_y$$



$\Rightarrow \ell_1'(t)=(0,1)\ ,\ \ell_2'(t)=(0,-1)$

- $\frac{1}{|\ell_1'(t)|}\cdot\ell_1'(t)_{\ell_1(t)} = (0,+1)_{(0,t)}$ – generates "upward" orientation
- $\frac{1}{|\ell_2'(t)|}\cdot\ell_2'(t)_{\ell_2(t)} = (0,-1)_{(0,-t)}$ – generates "downward" orientation

<u>Def.</u> An <u>oriented curve</u> is a curve along with a choice of orientation.

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ – circle

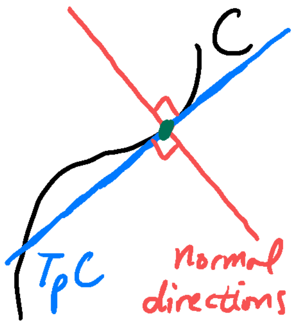
2 oriented curves made from $\mathcal{C}$

- anticlockwise ($\mathcal{C}_+$), clockwise ($\mathcal{C}_-$)

Next, restrict to case $n=2$ (curve $C \subseteq \mathbb{R}^2$)

- T_pC – 1-dimensional line in 2-dimensional plane
  (1-d subspace of $T_p\mathbb{R}^2$)

$\Rightarrow$ one remaining direction perpendicular to T_pC
(1-d subspace of $T_p\mathbb{R}^2$ perpendicular to T_pC) ← "normal"



(*) Again, useful to have unit arrows.

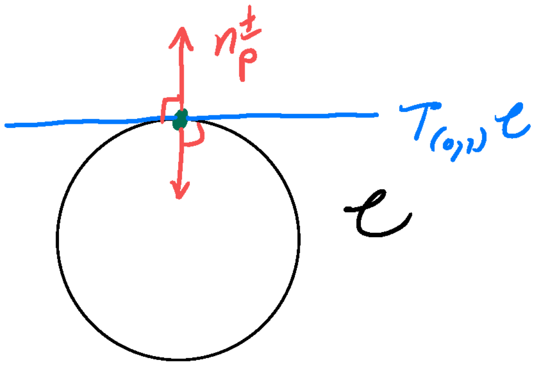
Def. Let $C \subseteq \mathbb{R}^2$ be a curve, and let $p \in C$. Then, $n_p \in T_p\mathbb{R}^2$ is a unit normal to C at p iff:

- n_p is perpendicular to every element of T_pC.
- $|n_p| = 1$

Ex. $\mathcal{C}$ – unit circle, $p = (0,1)$

$T_{(0,1)}\mathcal{C} = \{ s \cdot (1,0)_{(0,1)} \mid s \in \mathbb{R}\}$

unit normals: $n_p^{\pm} = (0, \pm 1)_{(0,1)}$



Q. How to compute unit normals?

(1) From unit tangents (via rotation)

Thm: Let $C \subseteq \mathbb{R}^2$ be a curve, and let $p \in C$. If $\pm(v_1,v_2)_p \in T_pC$ are the unit tangents to C at p, then the unit normals to C at p are: $n_p^{\pm} = \pm(-v_2, v_1)_p$.

Pf: $[\pm(v_1,v_2)_p][\pm(-v_2,v_1)_p] = 0$, and

$|\pm(-v_2,v_1)_p| = |(v_1,v_2)_p| = 1$.

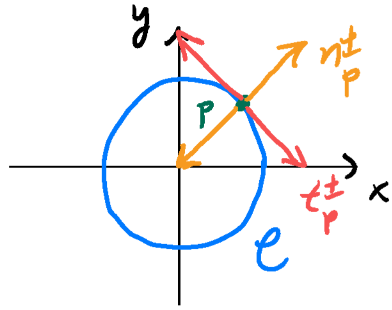
Remark. Where did this come from?

($\star$) $(-v_2, v_1)$ is 90° anticlockwise rotation of (v_1, v_2).

- $(v_1, v_2) \longmapsto v_1 + i v_2 \in \mathbb{C}$
- 90° rotation: $e^{\frac{\pi}{2} i}(v_1 + i v_2) = i(v_1 + i v_2) = -v_2 + i v_1$

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ - circle

$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



- Recall: unit tangents to $\mathcal{C}$ at p:

$$t_p^{\pm} = \pm\Big(\underbrace{-\tfrac{1}{\sqrt{2}}}_{v_1}, \underbrace{\tfrac{1}{\sqrt{2}}}_{v_2}\Big)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

$\Rightarrow$ unit normals to $\mathcal{C}$ at p: $n_p^{\pm} = \pm\Big(\underbrace{-\tfrac{1}{\sqrt{2}}}_{-v_2}, \underbrace{-\tfrac{1}{\sqrt{2}}}_{v_1}\Big)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

---

(2) Level Sets (more direct method when $\mathcal{C}$ is a level set)

Thm: Assume the setting of the "level set theorem":

- $\mathcal{C} = \{(x,y) \in U \mid f(x,y) = c\}$
- $U \subseteq \mathbb{R}^2$ - open, connected; $c \in \mathbb{R}$; $f : U \to \mathbb{R}$ - smooth
- $\nabla f(p) \neq \vec{0}_p$ for every $p \in C$

(In particular, C is a curve.)

Then, at any $p \in C$, the unit normals to C at p are

$$n_p^{\pm} = \pm \frac{1}{|\nabla f(p)|} \cdot \nabla f(p).$$

Remark: In particular, $\nabla f(p)$ points normal to its level set C.

Proof: Consider a parametrisation of C

$$\gamma : I \to C, \quad \gamma(t) = (x(t), y(t)).$$

Suppose $p = \gamma(t_0)$ $(t_0 \in I)$.

Then:
$$\begin{aligned} 0 &= \frac{d}{dt}[f(\gamma(t))] && (f(\gamma(t)) = c) \\ &= \partial_1 f(\gamma(t))\, x'(t) + \partial_2 f(\gamma(t))\, y'(t) && \text{(chain rule)} \\ &= \nabla f(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} \end{aligned}$$

At $p = \gamma(t_0)$: $\quad 0 = \pm \nabla f(p) \cdot [\underbrace{s \cdot \gamma'(t_0)_{\gamma(t_0)}}_{\in T_{\gamma(t_0)} = T_pC}] \qquad (s \in \mathbb{R})$

$\Rightarrow \pm \nabla f(p)$ normal to all elements of T_pC.

- To get unit normal, divide $\pm \nabla f(p)$ by its length.

Ex. Circle $\mathcal{C}$ satisfies

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid s(x,y) = 1\} \qquad \left(\begin{array}{l} s: \mathbb{R}^2 \to \mathbb{R} \\ s(x,y) = x^2 + y^2 \end{array} \right)$$

Again, take $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$\Rightarrow \nabla s(x,y) = (2x, 2y)_{(x,y)}$, $\quad \nabla s(p) = \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

- $|\nabla s(p)| = 2$

Thus, unit normals to $\mathcal{C}$ at p:

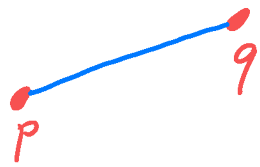
$$n_p^{\pm} = \pm \frac{1}{|\nabla s(p)|} \cdot \nabla s(p) = \pm \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

– Same as before!

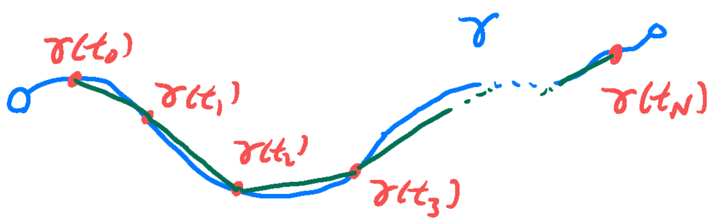
Previous geometric properties of curves – related to "shape".

Q. How to compute "size" of a curve.
length

Easy case: line segment ℓ.
length: $L(\ell) = |q - p|$



In general: consider parametric curve $\gamma: (a,b) \to \mathbb{R}^n$



Approximate γ by line segments

– Sample points
$\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_N)$
$a < t_0 < t_1 < \ldots < t_N < b$

Then: $$L(\gamma) \approx \sum_{k=1}^{N} |\gamma(t_k) - \gamma(t_{k-1})|$$

$$= \sum_{k=1}^{N} \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{|t_k - t_{k-1}|} \cdot \underbrace{\Delta t_k}_{= |t_k - t_{k-1}|}$$

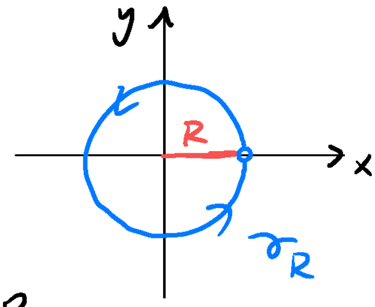
- To improve approximation, increase N, decrease Δt_k's.
- To find $L(\gamma)$ exactly, take "infinitely good approximation"

- "$N \nearrow \infty$", "$\Delta t_k \searrow 0$". (only informally here)

$$\text{"}\lim_{\substack{N\nearrow\infty \\ \Delta t \searrow 0}}\text{"} \sum_{k=1}^{N} \frac{|\gamma(t_k)-\gamma(t_{k-1})|}{|t_k - t_{k-1}|} \cdot \Delta t_k = \int_a^b |\gamma'(t)|\, dt$$

Def. Let $\gamma : (a,b) \to \mathbb{R}^n$ be a parametric curve. We define its <u>arc length</u> to be $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

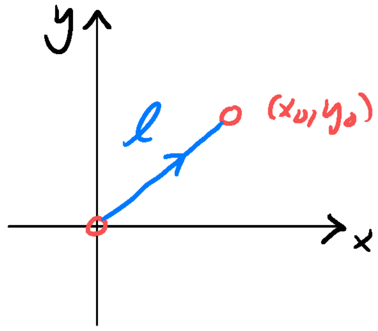
Ex. Let $R>0$, and
$\gamma_R : (0,2\pi) \to \mathbb{R}^2$, $\gamma_R(t) = (R\cos t, R\sin t)$
(circle of radius R, 1 revolution)



- $\gamma_R'(t) = (-R\sin t, R\cos t)$ • $|\gamma_R'(t)| = R$

Then, $L(\gamma_R) = \int_0^{2\pi} |\gamma_R'(t)|\, dt = \int_0^{2\pi} R\, dt = \underline{2\pi R}$.

Ex. Let $(x_0, y_0) \in \mathbb{R}^2$, and
$\ell : (0,1) \to \mathbb{R}^2$, $\ell(t) = (tx_0, ty_0)$
(line segment from $(0,0)$ to (x_0,y_0))



- $\ell'(t) = (x_0, y_0)$, • $|\ell'(t)| = \sqrt{x_0^2 + y_0^2}$

$\Rightarrow L(\ell) = \int_0^1 \sqrt{x_0^2+y_0^2}\, dt = \underline{\sqrt{x_0^2+y_0^2}}$.

<u>Recall</u>: For single integrals:

$\int_a^b 1\, dx = \text{Length of } (a,b)$, $\int_a^b f(x)\,dx =$ "weighted length"

Q. Can we extend this to parametric curves? ($\gamma : (a,b) \to \mathbb{R}^n$)

<u>Want</u>: • "$\int_\gamma 1\, ds$" $= L(\gamma) = \int_a^b |\gamma'(t)|\, dt$

• "$\int_\gamma F\, ds$" – "weighted length" of γ.

$$\int_a^b \underbrace{F(\gamma(t))}_{\substack{\text{weight} \\ \text{at } \gamma(t)}} \underbrace{|\gamma'(t)|}_{\text{speed of } \gamma \text{ at } \gamma(t).} dt$$

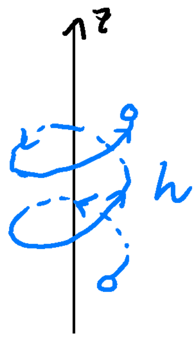
Def. Let $\gamma : (a,b) \to \mathbb{R}^n$ be a parametric curve, and let F be a real-valued function defined on the image of γ. The curve integral of F over γ is defined as

$$\int_\gamma F\, ds = \int_a^b F(\gamma(t))\, |\gamma'(t)|\, dt$$

(also called path or line integral)

Remark Note that $\int_\gamma 1\, ds = L(\gamma)$.

Ex: $h : (-2\pi, 2\pi) \to \mathbb{R}^3$, $h(t) = (\cos t, \sin t, t)$ – helix



$G : \mathbb{R}^3 \to \mathbb{R}$, $G(x,y,z) = x^2 + y^2 + z^2$

Find $\int_h G\, ds$.

Recall: • $h'(t) = (-\sin t, \cos t, 1)$ • $|h'(t)| = \sqrt{2}$

• $G(h(t)) = G(\cos t, \sin t, t) = 1 + t^2$

Thus, $$\int_h G\, ds = \int_{-2\pi}^{2\pi} G(h(t))\, |h'(t)|\, dt = \sqrt{2}\int_{-2\pi}^{2\pi} (1+t^2)\, dt$$

$$= \sqrt{2}\left(t + \tfrac{1}{3}t^3\right)\Big|_{t=-2\pi}^{t=2\pi} = \sqrt{2}\left(4\pi + \tfrac{2}{3}(2\pi)^3\right).$$

Q. Do these curve integrals define geometric properties of curves?

– Are they independent of parametrisation? (A. yes)

(*) Different particles travelling along the same path will go the same total distance.

Thm: Let $\gamma : (a,b) \to \mathbb{R}^n$ and $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^n$ be regular parametric curves. Assume γ and $\tilde{\gamma}$ are reparametrisations of each other. Also, let F be a real-valued function, defined on the images of γ and $\tilde{\gamma}$. Then:

$$\int_\gamma F\, ds = \int_{\tilde{\gamma}} F\, ds$$

• In particular, $L(\gamma) = L(\tilde{\gamma})$.

Proof: Let $\phi : (a,b) \longleftrightarrow (\tilde{a}, \tilde{b})$ be the change of variables:
$(\gamma(t) = \tilde{\gamma}(\phi(t)),\ t \in (a,b))$

$$\Rightarrow \ |\gamma'(t)| = |\phi'(t)|\,|\tilde{\gamma}'(\phi(t))|$$

Then: $$\int_\gamma F\,ds = \int_a^b F(\gamma(t))\,|\gamma'(t)|\,dt$$

$$= \int_a^b F(\tilde{\gamma}(\phi(t)))\,|\tilde{\gamma}'(\phi(t))|\,|\phi'(t)|\,dt$$

Substitution: $\tilde{t} = \phi(t)$, $d\tilde{t} = \phi'(t)\,dt$

$$= \int_{\tilde{a}}^{\tilde{b}} F(\tilde{\gamma}(\tilde{t}))\,|\tilde{\gamma}'(\tilde{t})|\,d\tilde{t}$$

$$= \int_{\tilde{\gamma}} F\,ds$$

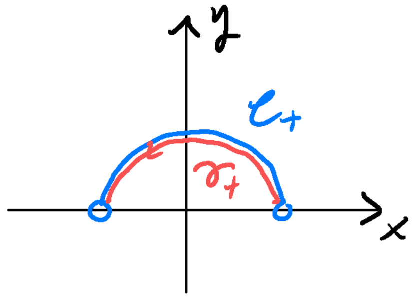
Thus, can make sense of integrals over curves.

(rather than just parametric curves)

Def. Let $C \subseteq \mathbb{R}^n$ be a curve, and let F be a real-valued function on C. Also, let $\gamma : I \to C$ be any injective parametrisation of C, such that the image of γ differs from C by only a finite number of points. Then, we define:

- The curve integral of F over C: $\displaystyle\int_C F\,ds = \int_\gamma F\,ds$
- The arc length of C: $L(C) = L(\gamma)$.

Ex. $\mathcal{C}_+ = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1,\ y > 0\}$
- upper half circle.



$\gamma_+ : (0,\pi) \to \mathcal{C}_+$, $\gamma_+(t) = (\cos t, \sin t)$
- injective parametrisation of all of $\mathcal{C}_+$

Thus: $$L(\mathcal{C}_+) = L(\gamma_+) = \int_0^\pi \underbrace{|\gamma_+'(t)|}_{1}\,dt = \pi.$$