Formal Definition of Curves

Last time: principles for defining curves
[1] Described using regular parametric curves
[2] "Independent of parametrisation"
(2 parametric curves that reparametrise each other describe the same curve.)
[3] Cannot self-intersect.

Def. $C \subseteq \mathbb{R}^n$ is a (smooth) curve iff:
for any point $p \in C$, there exist
(i) an open subset $V \subseteq \mathbb{R}^n$, with $p \in V$, and
(ii) a regular injective parametric curve $\gamma : I \to C$,
Such that the following hold:
(a) γ is a bijection between I and $C \cap V$.
(b) Its inverse $\gamma^{-1} : C \cap V \to I$ is also continuous.

Q. So, what is the meaning of this?
(Definition draws on background from beyond this module
$\Longrightarrow$ won't discuss in detail – focus on connection to principles)

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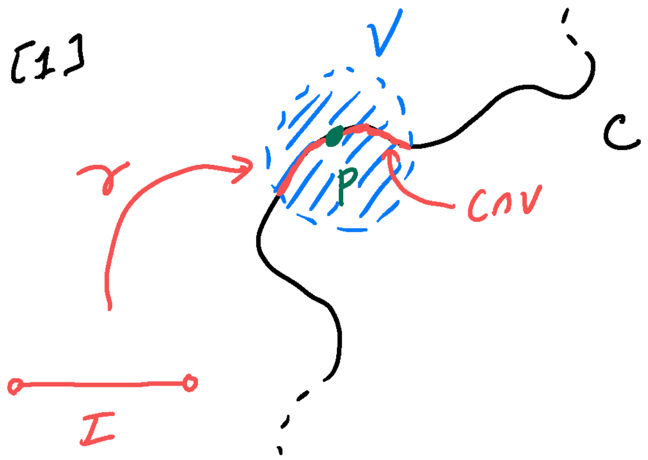
[1]



(*) Near p, the points of C look like a "deformed interval"
$C \cap V$

(*) Parametric curve γ:
- shows how interval I is deformed into $C \cap V$
- describes part of C

(*) C constructed by "gluing together" many deformed intervals

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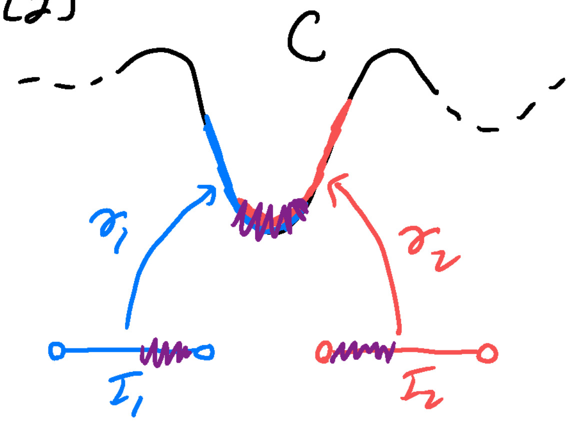
(*) Many possible descriptions γ of C near any $p \in C$
– None more "valid" than any other.

[2]

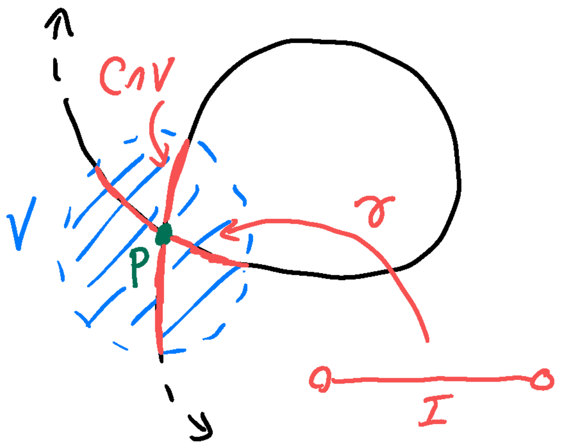


(※) (Thm.) In overlapping region (~~~), γ_1 and γ_2 are reparametrisations of each other.

- ~~~ described equally well by γ_1 and γ_2.

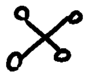
Q. What about self-intersections?

[3]



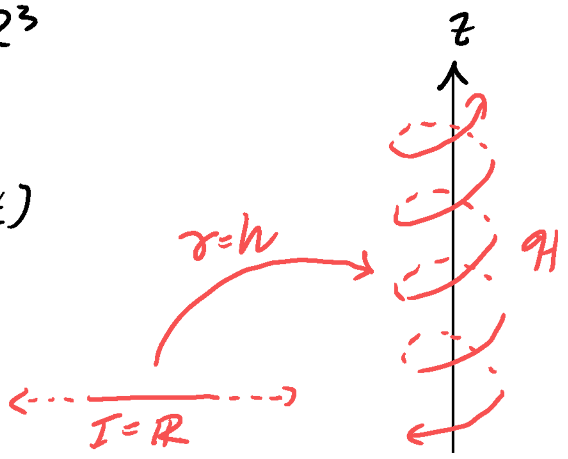
- γ $\underbrace{\text{continuous}}_{\text{no breaks}}$ and $\underbrace{\text{injective}}_{\text{no passing thru self}}$.

(※) But, cannot bend o—o into  without breaking or passing through self.

– "o—o and  have different topological structure"

Ex. $\mathcal{H} = \{(\cos t, \sin t, t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^3$
– helix

Let $h: \mathbb{R} \to \mathbb{R}^3$, $h(t) = (\cos t, \sin t, t)$
- regular parametric curve
- image of $h = \mathcal{H}$.
- h is injective



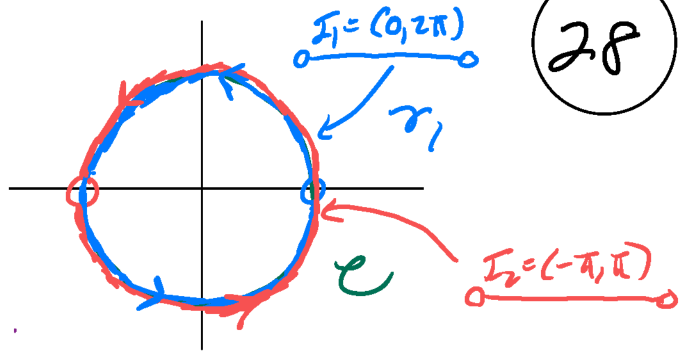
(※) Can show $\mathcal{H}$ is a curve $(V = \mathbb{R}^3,\ \gamma = h)$

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ – circle

- $\gamma_1 : (0, 2\pi) \to \mathbb{R}^2$, $\gamma_1(t) = (\cos t, \sin t)$
- $\gamma_2 : (-\pi, \pi) \to \mathbb{R}^2$, $\gamma_2(t) = (\cos t, \sin t)$

} injective, regular parametric curves

- γ_1 maps out $\mathcal{C} \setminus \{(1,0)\}$
- γ_2 maps out $\mathcal{C} \setminus \{(-1,0)\}$



(※) Neither γ_1, γ_2 describes all of $\mathcal{C}$
- $\mathcal{C}$ constructed by "gluing together" "deformed intervals" γ_1 and γ_2.

Remark: Can show $\mathcal{C}$ is a curve — see lecture notes.

Note: Definition of curves only allows for injective parametric descriptions
- In many cases, it's easier to work with non-injective descriptions.

Def: Let $C \subseteq \mathbb{R}^n$ be a curve. A parametrisation of C is any regular parametric curve $\gamma : I \to C$ (may not be injective).
(takes values only on C)

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ — circle.
$\gamma : \mathbb{R} \to \mathcal{C}$, $\gamma(t) = (\cos t, \sin t)$ — parametrisation of $\mathcal{C}$
- Image is all of $\mathcal{C}$.

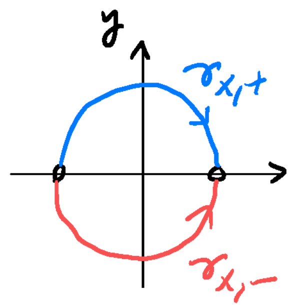
(※) Only need one parametrisation to describe $\mathcal{C}$, if injectivity not required.

Other parametrisations of $\mathcal{C}$:

- Take "$t = x$":

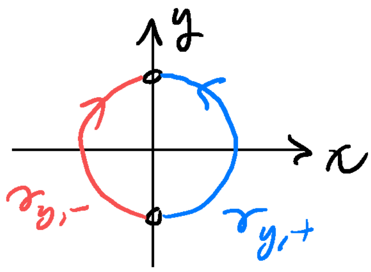
$\gamma_{x,+} : (-1,1) \to \mathcal{C}$, $\gamma_{x,+}(t) = (t, +\sqrt{1-t^2})$

$\gamma_{x,-} : (-1,1) \to \mathcal{C}$, $\gamma_{x,-}(t) = (t, -\sqrt{1-t^2})$



- Take "$t = y$"

$\gamma_{y,\pm} : (-1,1) \to \mathcal{C}$, $\gamma_{y,\pm}(t) = (\pm\sqrt{1-t^2}, t)$



good general strategies.

Q. Are there easier ways to show $C \subseteq \mathbb{R}^2$ is a curve?
(**A.** $n=2 \Rightarrow$ "good" level sets are curves.)

Thm: Let $U \subseteq \mathbb{R}^2$ be open and connected, and let $f: U \to \mathbb{R}$ be smooth. In addition, let $c \in \mathbb{R}$, and let
$$C = \{(x,y) \in U \mid f(x,y) = c\}$$ ~ "C is a level set of f"
If $\nabla f(p)$ is nonzero for every $p \in C$, then C is a curve.

(• Proof relies on implicit function theorem
• Can always parametrise via $\underbrace{x=t}_{\text{when } \partial_y f \neq 0}$ or $\underbrace{y=t}_{\text{when } \partial_x f \neq 0}$)

Ex. $S: \mathbb{R}^2 \to \mathbb{R}$, $S(x,y) = x^2+y^2$

Observe: unit circle $\mathscr{C}$ is a level set of S:
$$\mathscr{C} = \{(x,y) \in \mathbb{R}^2 \mid S(x,y) = 1\}$$

• $\nabla S(x,y) = (2x, 2y)_{(x,y)}$ $= (0,0)_{(x,y)}$ only when $(x,y) = (0,0)$

Since $(0,0) \notin \mathscr{C} \Rightarrow \nabla S(p)$ nonzero for all $p \in \mathscr{C}$.

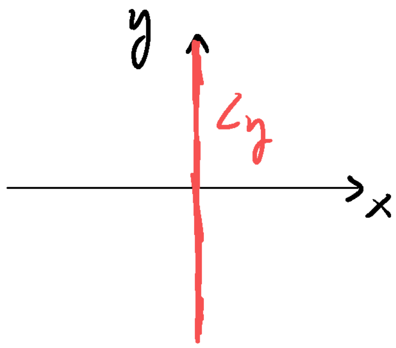
$\Rightarrow$ By theorem, $\mathscr{C}$ is a curve.

Ex. The y-axis L_y is a curve.

• $L_y = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{x}_{h(x,y)} = 0\}$

• $\nabla h(x,y) = (1,0)_{(x,y)} \neq (0,0)_{(x,y)}$.

$\Rightarrow$ L_y is a curve.



Note: $l_y: \mathbb{R} \to L_y$, $l_y(t) = (0,t)$ } injective parametrisation of L_y / image of $l_y = L_y$

Special Case: graphs of functions are curves.

Thm: Let I be an open interval, and let $h: I \to \mathbb{R}$ be smooth. Then, both of the following sets are curves:

graph of h – $\left(G_h = \{(t,h(t)) \mid t\in I\},\ G_h^* = \{(h(t),t) \mid t\in I\}.\right.$

Pf: (Only for G_h – proof for G_h^* is similar.)

Main idea: G_h is a level set

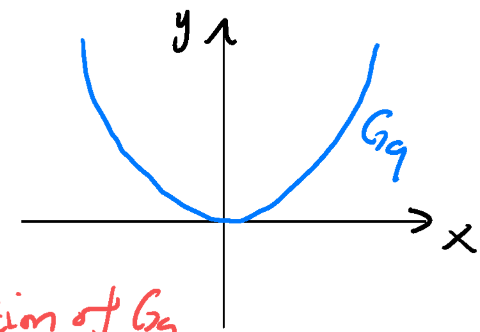
$$G_h = \{(x,y)\in I\times\mathbb{R} \mid \underbrace{y-h(x)}_{g(x,y)} = 0\}$$

• $\nabla g(x,y) = (-h'(x),1)_{(x,y)} \neq (0,0)_{(x,y)}$ for all $(x,y)\in I\times\mathbb{R}$.

Ex. $q:\mathbb{R}\to\mathbb{R}$, $q(x)=x^2$.

• Graph of q is a curve:

$$G_q = \{(t, \underbrace{t^2}_{q(t)}) \mid t\in\mathbb{R}\}$$



Note: $\left.\begin{array}{l}\gamma_q:\mathbb{R}\to G_q\\ \gamma_q(t)=(t,t^2)\end{array}\right\}$ injective parametrisation of G_q
image of $\gamma_q = G_q$

Remark: We have described different ways to describe curves:

• Parametric • Level set • Graph

Geometric Properties of Curves

Usually, study a curve C indirectly through parametrisations $\gamma: I\to C$.

Q. Does a given property of γ describe the geometry of C itself?

Ex. γ' – velocity – property of γ

– But not property of C.

(*) If given only $\underbrace{C}_{\text{path}}$ (and not $\underbrace{\gamma}_{\text{particle along path}}$ in particular), then cannot recover $\underbrace{\gamma'}_{\text{velocity of particle}}$.

Idea: To be a geometric property of C itself, the value extracted from any parametrisation of C should be the same.

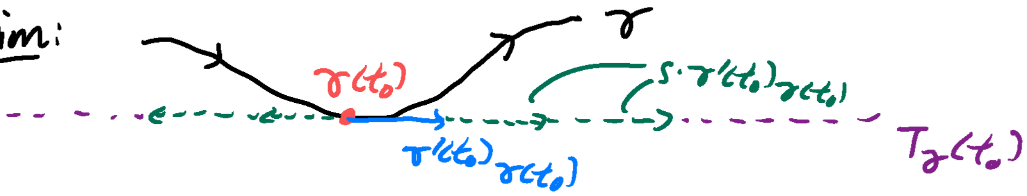
– [Z] "Independence of parametrisation"

Consider the following "modification" of velocity:

Def: Let $\gamma : I \to \mathbb{R}^n$ be a parametric curve, and let $t_0 \in I$. We define the *tangent line* to γ at t_0 by

$$T_\gamma(t_0) = \{ \underbrace{s \cdot \gamma'(t_0)_{\gamma(t_0)}}_{\text{arrows starting from } \gamma(t_0)} \mid s \in \mathbb{R} \} \quad (\subseteq T_{\gamma(t_0)}\mathbb{R}^n)$$

Interpretation:



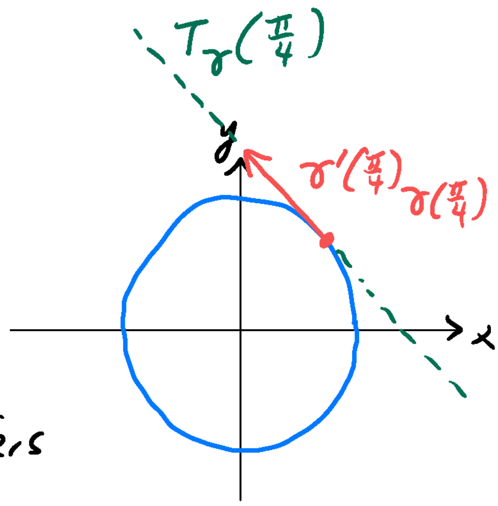
Remark: $T_\gamma(t_0) = \operatorname{span}\{\gamma'(t_0)_{\gamma(t_0)}\}$
– vector subspace of $T_{\gamma(t_0)}\mathbb{R}^n$.

Ex. $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$

$t_0 = \frac{\pi}{4} \implies \gamma(\frac{\pi}{4}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

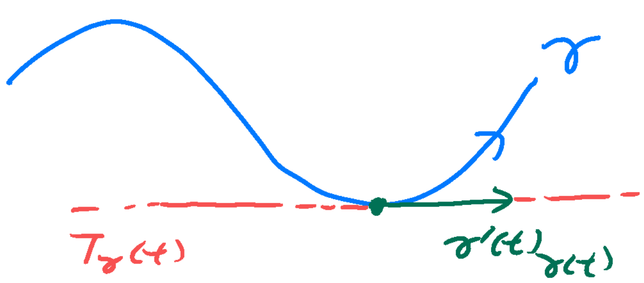
$\gamma'(\frac{\pi}{4}) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

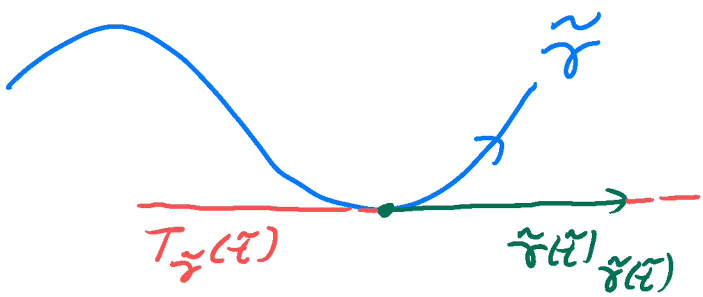
$$T_\gamma(\tfrac{\pi}{4}) = \{ s \cdot (-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} \mid s \in \mathbb{R} \}$$



Remark: Our definition of tangent line differs from what you have seen.
– Same interpretation, but emphasises direction and vector space structure.

Suppose $\gamma : I \to \mathbb{R}^n$ and $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$ are reparametrisations of each other. (γ, $\tilde{\gamma}$ have same directions, different speeds.)





(*) Tangent line – preserves directional information,
filters out speed information.

Thm: Let $\gamma: I \to \mathbb{R}^n$, $\tilde{\gamma}: \tilde{I} \to \mathbb{R}^n$ be regular parametric curves.
Assume γ is a reparametrisation of $\tilde{\gamma}$, with change of variables $\phi: I \leftrightarrow \tilde{I}$. $\quad (\gamma(t) = \tilde{\gamma}(\phi(t)).)$

Then, $T_\gamma(t) = T_{\tilde{\gamma}}(\phi(t))$.

Proof: Let $\tilde{t} = \phi(t) \implies \tilde{\gamma}(\tilde{t}) = \gamma(t)$ (definition)

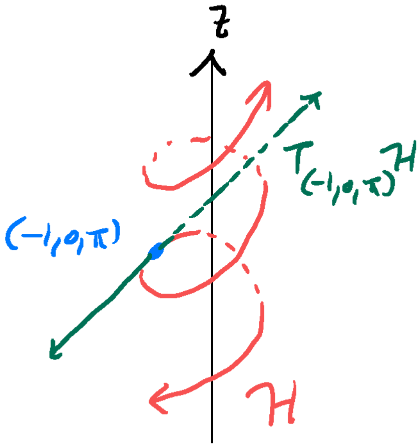
$\phi'(t)\,\tilde{\gamma}'(\tilde{t}) = \gamma'(t)$ (previous theorem)

$$T_\gamma(t) = \{s \cdot \gamma'(t)_{\gamma(t)} \mid s \in \mathbb{R}\} = \{s \cdot \tilde{\gamma}'(\tilde{t})_{\tilde{\gamma}(\tilde{t})} \mid s \in \mathbb{R}\} = T_{\tilde{\gamma}}(\tilde{t}).$$

Thus, tangent lines independent of parametrisation.
– Can define at level of curves.

Def: Let $C \subseteq \mathbb{R}^n$ be a curve, and let $p \in C$.
The tangent line to C at p is $T_pC = T_\gamma(t_0)$,
where γ is any parametrisation of C, with $\gamma(t_0) = p$.

Ex. $\mathcal{H} = \{(\cos t, \sin t, t) \mid t \in \mathbb{R}\}$ – helix
Find $T_{(-1,0,\pi)}\mathcal{H}$.



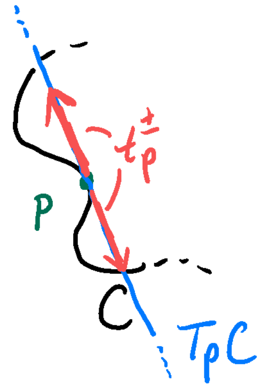
Recall: $h: \mathbb{R} \to \mathcal{H}$, $h(t) = (\cos t, \sin t, t)$
– parametrisation of $\mathcal{H}$

- $(-1, 0, \pi) = h(\pi)$
- $h'(t) = (-\sin t, \cos t, 1)$ $\quad$ • $h'(\pi) = (0, -1, 1)$

$$\implies T_{(-1,0,\pi)}\mathcal{H} = T_h(\pi) = \{s \cdot (0,-1,1)_{(-1,0,\pi)} \mid s \in \mathbb{R}\}$$

Observe: T_pC is a 1-dimensional vector space ($\simeq$ a "line")
- Thus, there are 2 elements of T_pC having unit length.

Def. Let $C \subseteq \mathbb{R}^n$ be a curve, and let $p \in C$. The <u>unit tangents</u> to C at p are the two elements $t_p^\pm \in T_pC$ satisfying $|t_p^\pm| = 1$.



<u>Interpretation</u>:
- $t_p^\pm$ point in opposite directions.
- Represent two directions you can go along C while standing at p.

Thm: Let $C \subseteq \mathbb{R}^n$ be a curve, let $\gamma : I \to C$ be a parametrisation of C, and let $t_0 \in I$. Then, the unit tangents to C at $p = \gamma(t_0)$ are

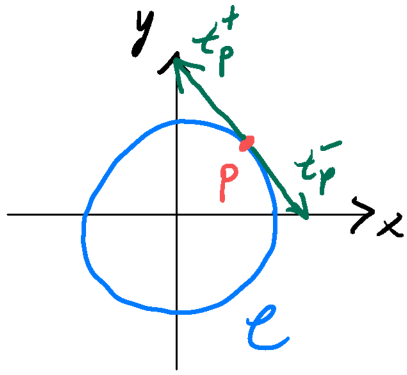
$$\underbrace{\pm \frac{1}{|\gamma'(t_0)|} \cdot \underbrace{\gamma'(t_0)_{\gamma(t_0)}}_{\text{in } T_pC}}_{\text{unit tangent vector}} .$$

Ex. $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ – <u>circle</u>

$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

<u>Recall</u>: $\gamma : \mathbb{R} \to \mathcal{C}$, $\gamma(t) = (\cos t, \sin t)$

$\gamma(\frac{\pi}{4}) = p$



Unit tangents at p:

$$t_p^\pm = \pm \underbrace{\frac{1}{|\gamma'(\frac{\pi}{4})|}}_{1} \overbrace{\gamma'(\tfrac{\pi}{4})}^{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{}_{\underbrace{\gamma(\frac{\pi}{4})}_{p}} = \pm\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} .$$