Differentiation of Vector-Valued Functions

- First, functions of one variable.

Def. Let $I = (a,b)$ (open interval), and let $f : I \to \mathbb{R}^n$.
The derivative of f at $t_0 \in I$ is

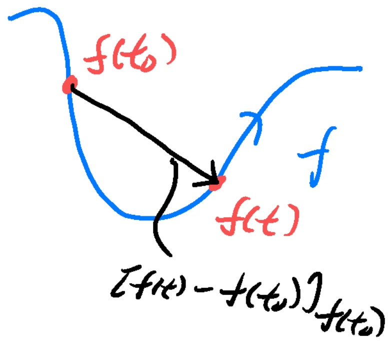
$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{(if exists)}$$

- Also, when $f'(t)$ exists for all $t \in I$, then we define the derivative of f t. be the function $f' : I \to \mathbb{R}$ mapping t to $f'(t)$.

One interpretation (classical mechanics):

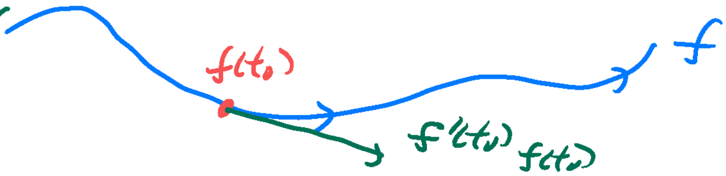
- $f(t) \in \mathbb{R}^n$ – position of particle at time t.
- $t - t_0$ – time elapsed.
- $f(t) - f(t_0)$ – change in position between times t_0, t (displacement)
- $\dfrac{f(t) - f(t_0)}{t - t_0}$ – average rate of change in position (time t_0 to t)

$\downarrow$ limit $t \to t_0$ $\downarrow$



- $f'(t_0)$ – instantaneous rate of change in position at time t_0 (velocity).

naturally drawn as arrows:



Q. How to compute derivatives?
- **A.** Componentwise.

Thm: Let f be as before, and write f in terms of components:

$$f(t) = (f_1(t), \ldots, f_n(t)) \qquad (f_k : I \to \mathbb{R})$$

Then, for any $t \in I$, $f'(t) = (f_1'(t), \ldots, f_n'(t))$
(as long as $f_1'(t), \ldots, f_n'(t)$ all exist).

Pf: $\dfrac{f(t)-f(t_0)}{t-t_0} = \left(\dfrac{f_1(t)-f_1(t_0)}{t-t_0}, \dots, \dfrac{f_n(t)-f_n(t_0)}{t-t_0}\right)$

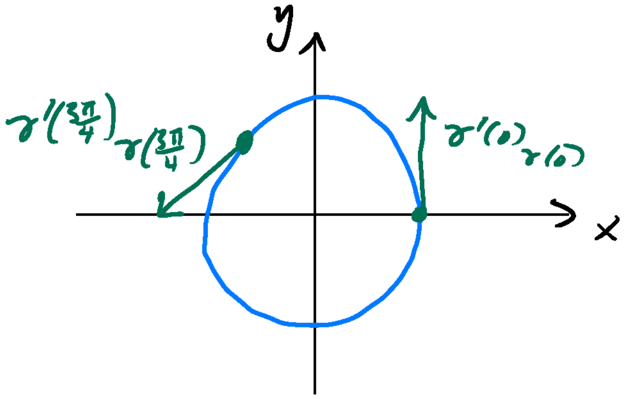
$\downarrow$ $t \to t_0$ $\downarrow$

$f'(t_0) = (f_1'(t_0), \dots, f_n'(t_0))$

Ex. $\gamma: \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$

$\Rightarrow \gamma'(t) = \left(\frac{d}{dt}\cos t, \frac{d}{dt}\sin t\right)$

$= \underline{(-\sin t, \cos t)}$



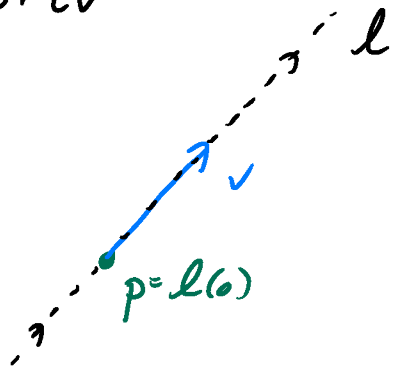
- $\gamma(0) = (1,0)$, $\gamma'(0) = (0,1)$
- $\gamma(\frac{3\pi}{4}) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\gamma'(\frac{3\pi}{4}) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Ex. Fix $p, v \in \mathbb{R}^n$. Define $\ell: \mathbb{R} \to \mathbb{R}^n$, $\ell(t) = p + tv$

- $\underline{\ell(0) = p}$
- Let $p = (p_1, \dots, p_n)$, $v = (v_1, \dots, v_n)$

$\Rightarrow \ell(t) = (p_1 + tv_1, \dots, p_n + tv_n)$

$\Rightarrow \underline{\ell'(t) = (v_1, \dots, v_n) = v}$



Next: functions of several variables

$f: U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$

Q. What should we assume for U to make sense of derivatives of f.

- Recall: Derivative at p $\approx$ "rate of change" at p.
  - Need: values of f "near p".

$\Rightarrow$ points "near p" must be in U (domain of f)

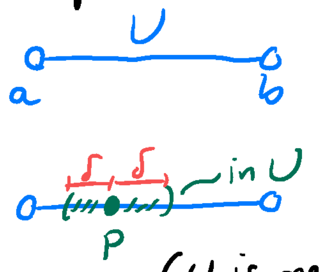
Def: $U \subseteq \mathbb{R}^m$ is open iff for any $p \in U$, there exists some $\underbrace{\delta > 0}_{\text{"sufficiently close"}}$ such that if $\underbrace{q \in \mathbb{R}^m \text{ and } |q-p| < \delta}_{\text{"q sufficiently close to p"}}$, then $q \in U$

~ "Any point sufficiently close to $p \in U$ also lies in U."

<u>Remark:</u> For those who took MTH5104 (Convergence & Continuity)

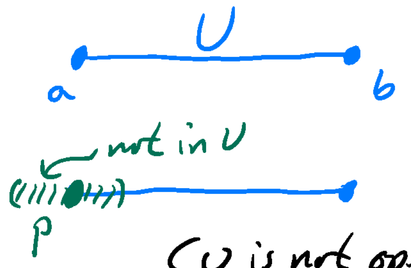
U is open $\iff \forall p \in U \; \exists \delta > 0 \; \forall q \in \mathbb{R}^m \; (|q-p| < \delta \rightarrow q \in U)$

<u>Ex.</u> $U = (a,b) \subseteq \mathbb{R}^1$
(open interval)



(U is open)

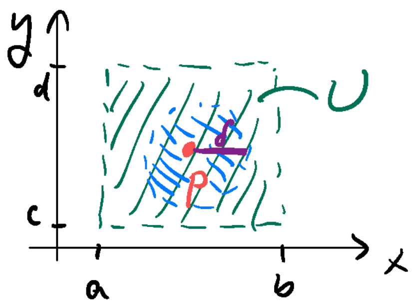
$U = [a,b] \subseteq \mathbb{R}^1$
(closed interval)



(U is not open)

<u>Ex.</u> $U = \underbrace{(a,b) \times (c,d)}_{} \subseteq \mathbb{R}^2$

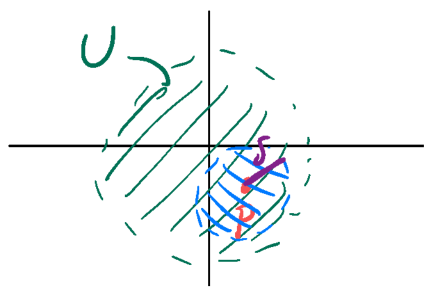
$\{(x,y) \mid x \in (a,b) \text{ and } y \in (c,d)\}$
(rectangle)



(U is open)

$U = \underbrace{B(0,1)}_{} \subseteq \mathbb{R}^2$

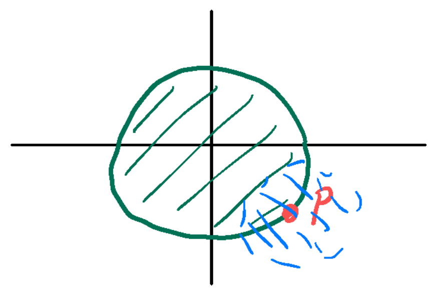
$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$
(open disk)



(U is open)

$U = \underbrace{\bar{B}(0,1)}_{} \subseteq \mathbb{R}^2$

$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
(closed disk)



(U is not open)

<u>Ex.</u> Can generalise to higher dimensions:

- Let $p \in \mathbb{R}^m$, $r > 0$
- $B(p,r) = \{q \in \mathbb{R}^m \mid |q-p| < r\}$ – open ("open ball")
- $\bar{B}(p,r) = \{q \in \mathbb{R}^m \mid |q-p| \leq r\}$ – not open ("closed ball")

For convenience, we want one further property for U.

<u>Motivation:</u> 1-d case – U = open interval (a,b)

- U is open

- U is "connected"

  if $x, y \in U$ and $x \le y$, then $[x, y] \subseteq U$

Why connected? If $U =$  ,

then $f : U \to \mathbb{R}^n$ can be treated as 3 independent functions
$f_1 : I_1 \to \mathbb{R}^n$, $f_2 : I_2 \to \mathbb{R}^n$, $f_3 : I_3 \to \mathbb{R}^n$

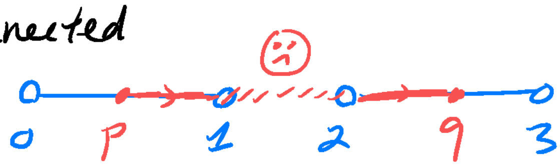
Now: want similar property in higher dimensions

- Issue: many ways to join points.

Def: An open subset $U \subseteq \mathbb{R}^m$ is connected iff for any $p, q \in U$, there is a path, lying entirely in U, going from p to q.
See lecture notes for definition

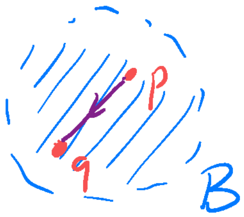
Ex.: • $I = (0,1)$ – connected

• $J = (0,1) \cup (2,3)$ – not connected

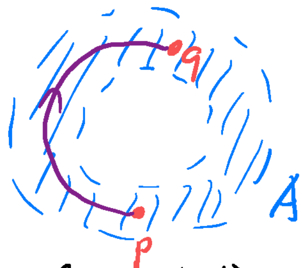


Remark: In fact, the open and connected subsets of $\mathbb{R}$ are precisely the open intervals.

Ex.



(connected)



(connected)

(not connected)

Finally: $f : U \to \mathbb{R}^n$ – what to assume for f to make sense of derivatives

- Want: U open and connected.

Differentiation in Several Variables

(derivative $\to$ partial derivative)

Def. Let $U \subseteq \mathbb{R}^m$ be open and connected, let $f: U \to \mathbb{R}^n$, and let $p = (p_1, \ldots, p_m) \in U$.

- For $1 \le k \le m$, we define the partial derivative in the k-th component of f at p by

$$\partial_k f(p) = \lim_{q \to p_k} \frac{f(p_1, \ldots, \overset{k\text{-th component}}{q}, \ldots, p_m) - f(p_1, \ldots, \hat{p_k}, \ldots, p_m)}{q - p_k} \qquad \left(\begin{matrix}\text{if}\\ \text{exists}\end{matrix}\right)$$

- Also, if $\partial_k f(p)$ exists for all $p \in U$, then we can define $\partial_k f: U \to \mathbb{R}^n$, mapping p to $\partial_k f(p)$.

Remark: When $m = 1$, then $\partial_1 f = f'$.

Ex. $h: \mathbb{R}^2 \to \mathbb{R}$ $\quad h(u,v) = u + v^2 + u^3 v$

- To find $\partial_1 h(u,v)$ – hold v constant, differentiate in u:
  - $\partial_1 h(u,v) = 1 + 0 + 3u^2 v = \underline{1 + 3u^2 v}$
  - $\partial_1 h(1,1) = 1 + 3 \cdot 1^2 \cdot 1 = \underline{4}$
- To find $\partial_2 h(u,v)$ – hold u constant, differentiate in v:
  - $\partial_2 h(u,v) = 0 + 2v + u^3 = \underline{2v + u^3}$
  - $\partial_2 h(1,1) = 2 \cdot 1 + 1^3 = \underline{3}$

Q. How to compute partial derivatives of vector-valued functions?

Thm: Let f be as before, and write $f(p) = (f_1(p), \ldots, f_n(p))$ $\quad (f_k: U \to \mathbb{R})$

Then, for any $1 \le k \le m$ and $p \in U$:

$$\partial_k f(p) = (\partial_k f_1(p), \ldots, \partial_k f_n(p)) \qquad \text{(if right-hand side exists)}$$

(Pf: Same as theorem for derivatives – see lecture notes.)

Ex. $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$, $\quad \sigma(u,v) = (\cos u, \sin u, v)$ $\quad$ (cylinder)

- $\partial_1 \sigma(u,v) = \left(\frac{d}{du}\cos u, \frac{d}{du}\sin u, \frac{d}{du} v\right) = \underline{(-\sin u, \cos u, 0)}$

- $\partial_2\sigma(u,v) = \left(\frac{d}{dv}\cos u, \frac{d}{dv}\sin u, \frac{d}{dv}v\right) = (0,0,1)$

Q. How to interpret partial derivatives?
(Take previous example)

- Hold $v=v_0$ constant and vary u: $\gamma(u) = \sigma(u,v_0)$

$$\Rightarrow \gamma'(u_0) = \lim_{u\to u_0} \frac{\sigma(u,v_0)-\sigma(u_0,v_0)}{u-u_0} = \partial_1\sigma(u_0,v_0).$$

$\sim \partial_1\sigma$ - "derivative of σ obtained by holding v constant"

$\sim \partial_1\sigma(u_0,v_0)_{\sigma(u_0,v_0)} = \gamma'(u_0)_{\gamma(u_0)}$ - arrow on path along image of σ by holding v constant.

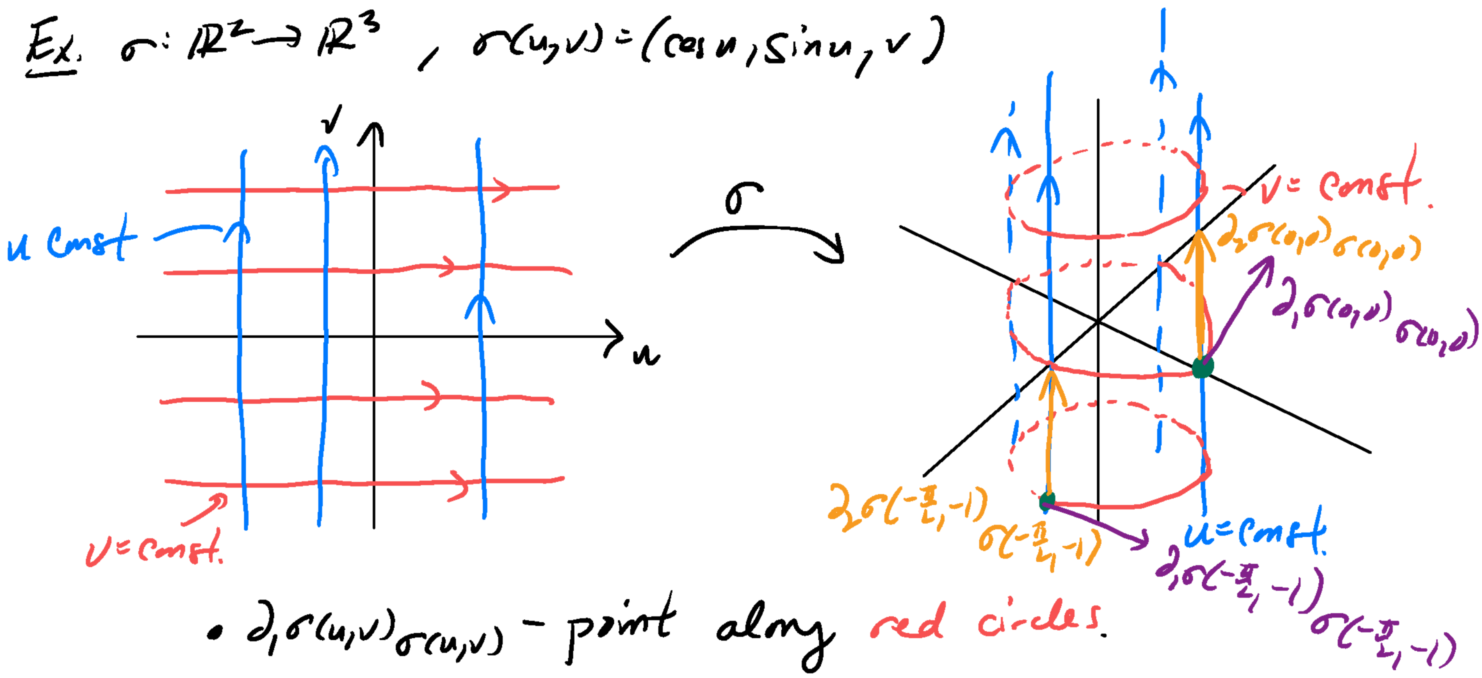
- Hold $u=u_0$ constant and vary v: $\lambda(v) = \sigma(u_0,v)$

$$\Rightarrow \lambda'(v_0) = \partial_2\sigma(u_0,v_0)$$

$\sim \partial_2\sigma$ - "derivative of σ obtained by holding u constant"

$\sim \partial_2\sigma(u_0,v_0)_{\sigma(u_0,v_0)} = \lambda'(v_0)_{\lambda(v_0)}$ - arrow on path along image of σ by holding u constant

Ex. $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$



- $\partial_1\sigma(u,v)_{\sigma(u,v)}$ - point along red circles.
- $\partial_2\sigma(u,v)_{\sigma(u,v)}$ - point along blue lines.

For instance:

- $\sigma(0,0) = (1,0,0)$
- $\partial_1\sigma(0,0) = (0,1,0)$
- $\partial_2\sigma(0,0) = (0,0,1)$
- $\sigma(-\frac{\pi}{2},-1) = (0,-1,-1)$
- $\partial_1\sigma(-\frac{\pi}{2},-1) = (1,0,0)$
- $\partial_2\sigma(-\frac{\pi}{2},-1) = (0,0,1)$

Vector fields) but different approach than before

↳ "Functions whose values are tangent vectors."

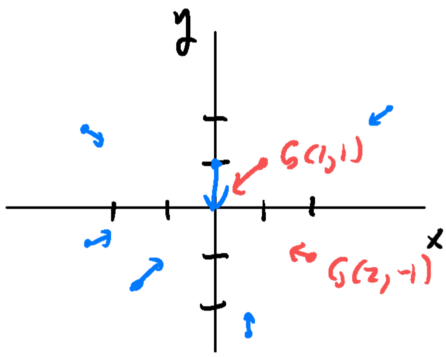
Def Let $A \subseteq \mathbb{R}^n$. A vector field on A is a function F mapping each $p \in A$ to a tangent vector $F(p) \in T_p\mathbb{R}^n$

(∴ $F(p)$ is an "arrow" starting at p.) ~ different from calculus and differential equations.

- $n \leq 3$: natural way to visualise vector fields ~ draw the arrows!

Ex. Vector field G on $\mathbb{R}^2 \setminus \{(0,0)\}$ $(n=2)$

$$G(p) = -\frac{1}{|p|^3} \cdot p_p$$



- $G(1,1) = -\frac{1}{2\sqrt{2}}(1,1)_{(1,1)} = \left(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)_{(1,1)}$
- $G(2,-1) = -\frac{1}{5\sqrt{5}} \cdot (2,-1)_{(2,-1)} = \left(\frac{-2}{5\sqrt{5}}, \frac{1}{5\sqrt{5}}\right)_{(2,-1)}$
  - $G(p)$ - points toward origin
  - $|G(p)| = \frac{1}{|p|^2}$ ~ larger when $|p|$ is smaller – arrows longer near origin.

G (when $n=3$) represents Newtonian gravitational force from object at origin

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One more familiar concept from calculus: gradients (but define a bit differently here)

Def. Let $U \subseteq \mathbb{R}^m$, and let $f : U \to \mathbb{R}$ (real-valued!) The gradient of f at $p \in U$ is defined to be

$$\nabla f(p) = (\partial_1 f(p), \dots, \partial_n f(p))_p \in T_p\mathbb{R}^m \quad \text{(if exists)}$$

- If $\nabla f(p)$ exists for every $p \in U$, then we define the gradient of f to be the vector field on U mapping each p to $\nabla f(p) \in T_p\mathbb{R}^m$.

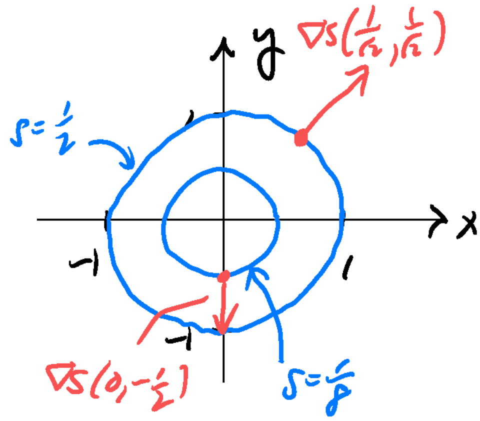
<u>Remark</u>: $\nabla f(p)$ interpreted as "arrow" starting from p
- Different from calculus definition (as "vector")
- But same intuition for both definitions.

<u>Ex.</u> $s : \mathbb{R}^2 \to \mathbb{R}$, $s(x,y) = \frac{1}{2}(x^2+y^2)$

$\Rightarrow \partial_1 s(x,y) = x$, $\partial_2 s(x,y) = y$

$\Rightarrow \nabla s(x,y) = (\partial_1 s(x,y), \partial_2 s(x,y))_{(x,y)} = (x,y)_{(x,y)}$

- $\nabla s\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$
- $\nabla s\left(0, -\frac{1}{2}\right) = \left(0, -\frac{1}{2}\right)_{\left(0, -\frac{1}{2}\right)}$



<u>Remark</u>: ∇s always perpendicular to the level sets $s = $ constant.
(will see why later)

<u>Ex.</u> $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $g(p) = \frac{1}{|p|}$ $\quad \left(g(x,y) = \frac{1}{\sqrt{x^2+y^2}} \right)$

- $\partial_1 g(x,y) = -\frac{1}{2}(x^2+y^2)^{-3/2} \, 2x = -\frac{x}{(x^2+y^2)^{3/2}}$
- $\partial_2 g(x,y) = -\frac{y}{(x^2+y^2)^{3/2}}$

$\Rightarrow \nabla g(x,y) = -\frac{1}{(x^2+y^2)^{3/2}} \cdot (x,y)_{(x,y)}$

~ in other words $\nabla g(p) = -\frac{1}{|p|^3} \cdot p_p = G(p)$ – gravitational force

- g – called <u>gravitational potential</u> in physics

<u>Remark</u>: G can be expressed as a gradient, hence has special properties (will hopefully discuss later in module).