MTH5113 (Week 1)

- What is this module?
  - Geometry - study of "shapes" and "sizes" of objects.
    - How to quantify and compute?
  - Previously (calculus, linear algebra) - flat, linear spaces ($\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^n$)
    - Here, extend to curved settings (e.g. line $\to$ circle)
      (e.g. flat paper $\to$ surface of ball)
  - Apply tools from calculus and linear algebra to study geometry.
  - Why? Many things in the real world are curved.
    - e.g. planetary motion, surface of earth, universe (relativity)

Module details

- Lectures: 4 hours / week
  - 3 hours "lecture"
  - 1 hour "tutorial"

  mixed together

  problems, examples
- Lecture material:
  - Typed lecture notes (QMPlus / Lecture Material)
  - Live lectures — interactive
  - Pre-recorded lectures (QMPlus / Lecture Material)

    from 2020/21 — polished, edited for errors
- Assessments:
  - 5 courseworks ($5 \times 4\%$)

    due W3, W5, W8, W10, W12

    ~ problem sheets weekly, 2 per coursework
  - Final exam (80%)
- Prerequisites:
  - Calculus (MTH4100/4200, MTH4101/4201)
  - Linear Algebra (MTH4115/4215, MTH5112/5212)
- What will you do?
  - Conceptual understanding
  - Computations
  - Visual understanding (graphs/plots)

  (- Minimal proofs)

- Module outline: (1) Revision – vectors, calculus
  (New geometric perspective on old things)
  (2) Geometry of curves (1-d curved objects)
  (3) Geometry of surfaces (2-d curved objects)
  (4) Topics/Applications (optimisation, Lagrange multipliers, extending fundamental theorem of calculus

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Vectors – What are they?

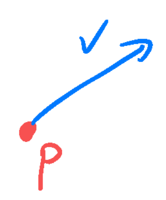
(1) Linear algebra – elements of $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}^n\}$ or vector space

here, usually $n = 1, 2, 3$

(2) Visually – "arrows"

Goal: Formally develop idea (2).

Q. How to represent "arrow" in $\mathbb{R}^n$?



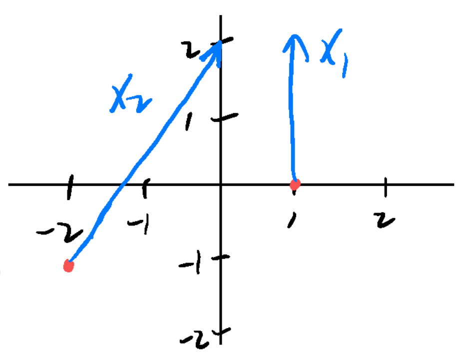
- Start point $p \in \mathbb{R}^n$
- "Vector component" $v \in \mathbb{R}^n$
  (Captures direction, length of "arrow")

Def. A tangent vector in $\mathbb{R}^n$ is a pair (v, p), usually written v_p here, where $v, p \in \mathbb{R}^n$.

(v_p – "arrow starting at p, pointing along v")

Ex. $X_1 = (0,2)_{(1,0)}$

$X_2 = (2,3)_{(-2,-1)}$



(X_1 – arrow from $(1,0)$ to $(1,0) + (0,2) = (1,2)$

X_2 – arrow from $(-2,-1)$ to $(-2,-1) + (2,3) = (0,2)$

Def. For any $p \in \mathbb{R}^n$, the tangent space at p is

$$T_p\mathbb{R}^n = \{v_p \mid v \in \mathbb{R}^n\}$$ ~ "set of all arrows starting from p"

Q. What geometrically meaningful things can we do with tangent vectors?

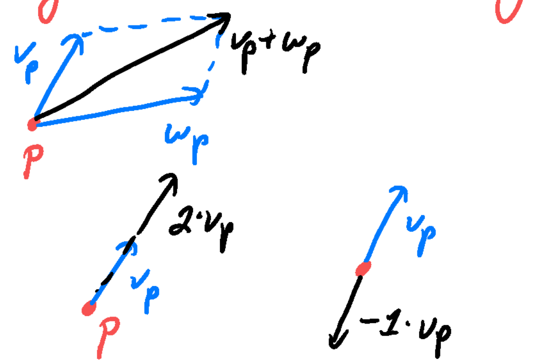
Def. Given $p \in \mathbb{R}^n$; $v_p, w_p \in T_p\mathbb{R}^n$; and $c \in \mathbb{R}$:

(1) Addition: $v_p + w_p = (v+w)_p$

(2) Scalar multiplication: $c \cdot v_p = (c \cdot v)_p$

apply usual ops to vector component.

Why do we define this way?



Ex. • $(1,2)_{(-1,0)} + (3,-1)_{(-1,0)} = [(1,2)+(3,-1)]_{(-1,0)}$
$= (4,1)_{(-1,0)}$

• $(0,1)_{(0,0)} + (1,0)_{(0,1)}$ – not defined!
not the same

• $-2 \cdot (1,2,3)_{(-1,-2,-3)} = [-2 \cdot (1,2,3)]_{(-1,-2,-3)} = (-2,-4,-6)_{(-1,-2,-3)}$

(*) $T_p\mathbb{R}^n$, with operations (1) and (2), form a n-dimensional vector space!

– "linear algebraic structure" associated with arrows

What other useful operations on "arrows" can we define?

(I) "Length" of arrows

Def. Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$

• The norm of v is $|v| = \sqrt{v_1^2 + \ldots + v_n^2}$

• The norm of v_p is defined as $|v_p| = |v|$

Ex. $|(1,2,-1,7)_{(0,0,3,0)}| = |(1,2,-1,7)| = \sqrt{1+4+1+49} = \sqrt{55}$

Remark • $|v_p|$ – "length of v_p"

• $|v_p| \neq 0 \Rightarrow \frac{1}{|v_p|} \cdot v_p$ ~ unit tangent vector in direction of v_p.

(II) Dot Products

Def. Let $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, and let $p \in \mathbb{R}^n$

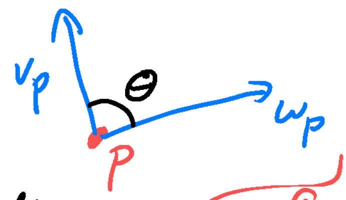
• Recall the dot product: $v \cdot w = v_1 w_1 + \ldots + v_n w_n \in \mathbb{R}$ – scalar!

- Dot product of arrows: $v_p \cdot w_p = v \cdot w \in \mathbb{R}$ – Scalar!

Ex. • $(1,2)_{(-1,-1)} \cdot (-1,2)_{(-1,-1)} = (1,2)\cdot(-1,2) = 1\cdot(-1) + 2\cdot 2 = \underline{3}$

• $(1,2)_{(-1,-1)} \cdot (-1,2)_{(-1,0)}$ – not defined!
not the same.

Q. Why define $v_p \cdot w_p$ this way?
(A. Has geometric meaning.)



Thm. Let v, w, p be as before. Then
$$v_p \cdot w_p = |v_p|\,|w_p| \cos\theta,$$
where θ is the angle between v_p and w_p at p.

(✱) Our definition of dot product captures the familiar picture.

Cor. $|v_p|^2 = v_p \cdot v_p$) angle between v_p and itself is $\theta = 0$, so $\cos\theta = 1$.

(✱) Dot product mixes two pieces of information
- Lengths of arrows involved.
- Angle between arrows involved.

(III) Cross products (in 3-d)

Def. Let $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ and $p \in \mathbb{R}^3$.

- Recall the cross product
$$v \times w = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1) \in \mathbb{R}^3 \text{ – vector!}$$
- Cross product of arrows: $v_p \times w_p = (v \times w)_p \in T_p\mathbb{R}^3$ – tangent vector!

Remark: Heuristic formula:
$$v \times w = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$
$\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$

Remark: Unlike dot products, only defined in 3-d!

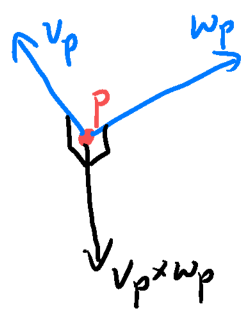
Q. Why define cross product this way?
(A. Has geometric meaning.)

<u>Thm</u> Let v, w, p be as before. Then:

- If v_p, w_p point in the same or opposite directions, then $v_p \times w_p = 0$.
- Otherwise, $v_p \times w_p$ points in direction perpendicular to v_p and w_p and satisfies "right-hand rule".

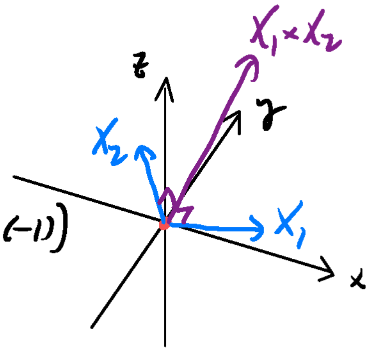
  Also $|v_p \times w_p| = |v_p|\,|w_p| \sin\theta$

  same as before



(*) Given vectors in 2 of 3 dimensions, cross product generates vector in 3rd dimension.

<u>Ex.</u> $X_1 = (1,1,-1)_{(0,0,0)}$, $X_2 = (-1,1,0)_{(0,0,0)}$



- $(1,1,-1) \times (-1,1,0) = (1\cdot 0 - (-1)\cdot 1,\ (-1)\cdot(-1) - 1\cdot 0,\ 1\cdot 1 - 1\cdot(-1))$
  $= (1,1,2)$

$\Rightarrow X_1 \times X_2 = [(1,1,-1)\times(-1,1,0)]_{(0,0,0)} = (1,1,2)_{(0,0,0)}$

<u>Vector-valued Functions</u>

To model more complex objects $\Rightarrow$ look at functions

- vector-valued functions : $f: A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$

  ~ Many ways to view f:

  - vector $x \in A$ $\to$ vector $f(x) \in \mathbb{R}^n$
    $\to$ n real values $f(x) = (f_1(x), \dots, f_n(x))$
    $(f_1, \dots, f_n : A \to \mathbb{R})$

  - m real values $x_1, \dots, x_m$ $\to$ vector $f(x_1, \dots, x_m)$
    $\to$ n real values $(f_1(x_1,\dots,x_m), \dots, f_n(x_1,\dots,x_m))$

(*) We will use all these styles, when convenient

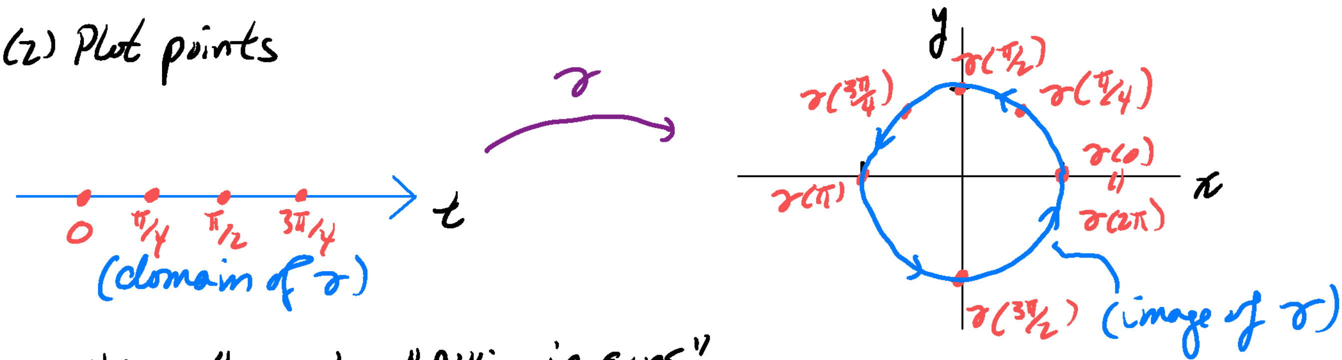
<u>Ex.</u> $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ $(m=1, n=2)$

<u>Q.</u> How to plot/visualise γ?

<u>Strategy:</u> (1) Compute $\gamma(t)$ for sample of t's.

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$3\pi/2$	2π
$\gamma(t)$	$(1,0)$	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(0,1)$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-1,0)$	$(0,-1)$	$(1,0)$

(2) Plot points



(3) "Guess" γ by "filling in gaps".

(☆) Image of γ is a <u>unit circle</u> (centre at $(0,0)$).
(– can also see from definitions of $\sin$, $\cos$.)

<u>Remark:</u> Plotting is similar when $n=3$ (harder to draw)
– Will not need to plot when $n \geq 4$.

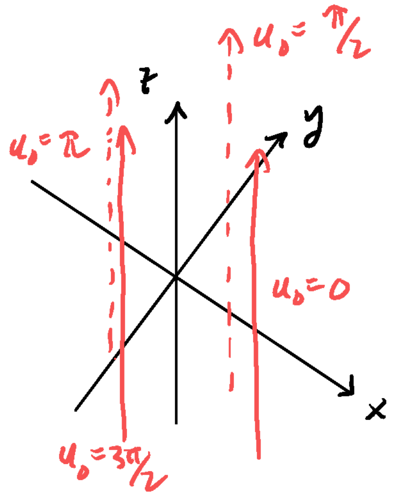
Next, consider functions of multiple variables:

<u>Ex.</u> $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$

<u>Q.</u> How to plot/visualise σ? $(m=2, n=3)$

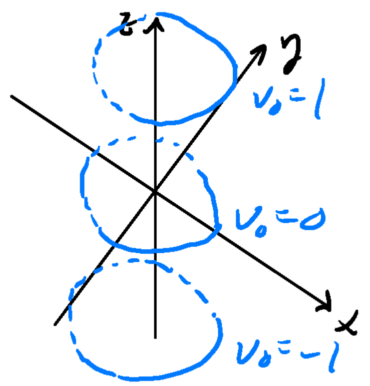
<u>Strategy</u> (1) Fix $u = u_0$, and vary v:
$\gamma(v) = \sigma(u_0, v) = (\cos u_0, \sin u_0, v)$
Plot for several u_0's.

upward vertical lines, at fixed polar angle.
$u_0 = 0$: $\sigma(0,v) = (1,0,v)$
$u_0 = \frac{\pi}{2}$: $\sigma(\frac{\pi}{2}, v) = (0,1,v)$
⋮

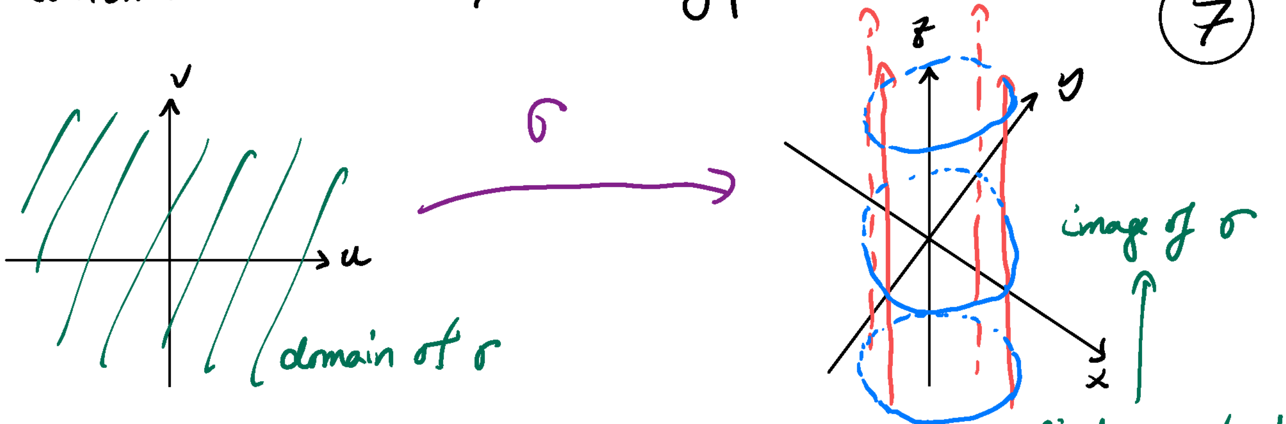


(2) Fix $v = v_0$, and vary u:
$\lambda(u) = \sigma(u, v_0) = (\cos u, \sin u, v_0)$

circles in x-y planes at fixed z-height
$v_0 = 0$: $\sigma(u,0) = (\cos u, \sin u, 0)$
$v_0 = 1$: $\sigma(u,1) = (\cos u, \sin u, 1)$
⋮



(3) Combine (1) and (2), "fill in gaps"



Interpretation: rolling (infinite) piece of paper into cylinder.

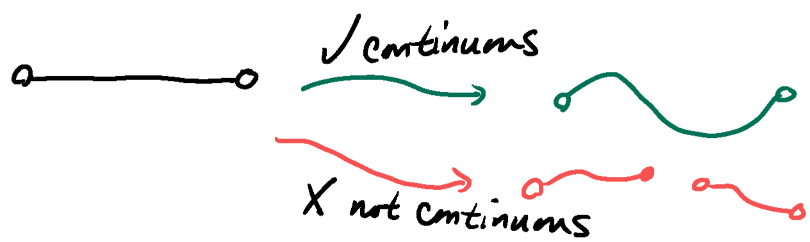
(Don't worry, will see many more examples.)

Limits and Continuity

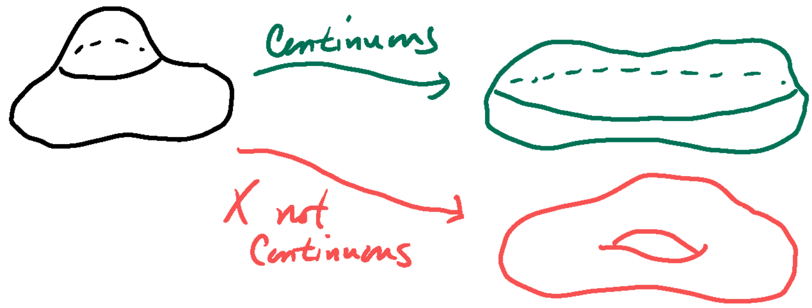
Next, an informal discussion of continuity for vector-valued functions.

- First step toward calculus of vector-valued functions.

Ex.



Ex.



Q. Precise definition of continuity?

- First, need to discuss a familiar concept - limits.

Def. Let $A \subseteq \mathbb{R}^m$, let $f: A \to \mathbb{R}^n$, and let $p \in A$.
L is the limit of f at p, denoted $\lim_{x \to p} f(x) = L$, iff for all $\varepsilon > 0$, there is some $\delta > 0$, such that:
if $x \in A$, $x \neq p$, and $|x - p| < \delta$, then $|f(x) - L| < \varepsilon$.

~ "$f(x)$ gets as close as you want to L, as long as x is close enough to p."

- ε – how close you want $f(x)$ to be to L.
- δ – how close x must be to p.

Q. How to compute limits of vector-valued functions?
(A. Componentwise)

Thm: Let $f: A \to \mathbb{R}^n$ be as before, write $f(x) = (f_1(x), \ldots, f_n(x))$, and let $p \in A$. Then, $\lim_{x\to p} f(x) = L = (L_1, \ldots, L_n) \in \mathbb{R}^n$ if and only if $\lim_{x \to p} f_k(x) = L_k$ for every $1 \le k \le n$.

"$\lim_{x\to p} f(x) = \left(\lim_{x\to p} f_1(x), \ldots, \lim_{x\to p} f_n(x)\right)$"

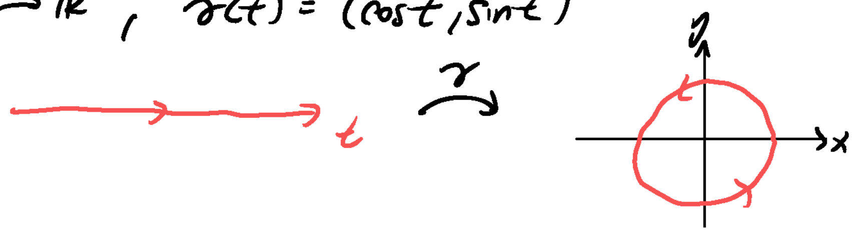
(Proof: see lecture notes.)

Def. Let $f: A \to \mathbb{R}^n$ be as before, and let $p \in A$.

- f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$
- Also, f is continuous iff f is continuous at every $x \in A$.

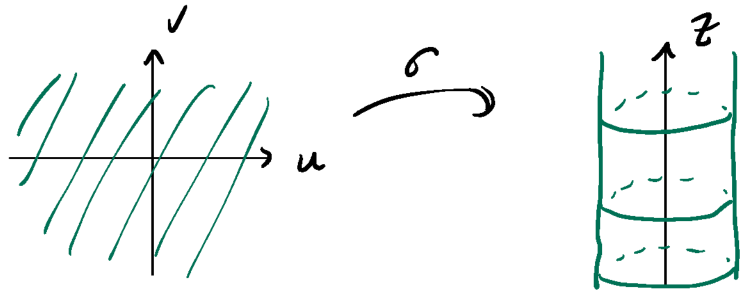
"values of f near p are close to $f(p)$"
↳ values of f do not "jump" or "break" at p.

Ex. $\gamma: \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$



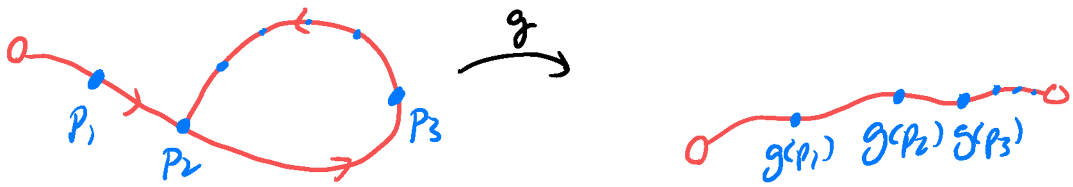
- γ "bends wire into circular shape"
  without breaking! ⟶ Continuous!
- nearby values on $\mathbb{R}$ map to nearby values on circle.

Ex. $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) = (\cos u, \sin u, v)$



- σ "rolls up paper into cylindrical shape"

Ex.



not continuous! $\sim$ g "breaks" figure at p_2

- As you travel along domain toward p_2
- Corresponding values of g do not go toward $g(p_2)$.