MTH5113 (Winter 2022): Problem Sheet 9
Solutions

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(1) *(Warm-up)*

(a) Consider the (real-valued) function

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = xy^2z^3,$$

as well as the parametric surface

$$\mathbf{P} : (0, 1) \times (0, 1) \to \mathbb{R}^3, \qquad \mathbf{P}(u, v) = (1,\ u,\ v).$$

Compute the surface integral of F over $\mathbf{P}$.

(b) Consider the (real-valued) function

$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = x^2 + y^2,$$

as well as the parametric surface

$$\tau : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3, \qquad \tau(u, v) = (v \cos u,\ v \sin u,\ v).$$

Compute the surface integral of G over τ.

(a) We begin by computing some required quantities. Differentiating $\mathbf{P}$ yields

$$\partial_1 \mathbf{P}(u, v) = (0,\ 1,\ 0), \qquad \partial_2 \mathbf{P}(u, v) = (0,\ 0,\ 1),$$

and their cross product satisfies

$$\partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v) = (1,\ 0,\ 0), \qquad |\partial_1 \mathbf{P}(u, v) \times \partial_2 \mathbf{P}(u, v)| = 1.$$

Furthermore, observe that

$$F(\mathbf{P}(u, v)) = F(1,\ u,\ v) = u^2v^3.$$

Thus, by the definition of the (parametric) surface integral, we obtain

$$\begin{aligned}
\iint_{\mathbf{P}} F \, dA &= \iint_{(0,1)\times(0,1)} F(\mathbf{P}(u,v)) |\partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v)| \, du dv \\
&= \int_0^1 \int_0^1 (u^2 v^3 \cdot 1) \, du dv \\
&= \int_0^1 u^2 \, du \int_0^1 v^3 \, dv \\
&= \frac{1}{12}.
\end{aligned}$$

(b) First, we differentiate τ,

$$\partial_1 \tau(u,v) = (-v \sin u, \, v \cos u, \, 0), \qquad \partial_2 \tau(u,v) = (\cos u, \, \sin u, \, 1),$$

and we compute their cross product:

$$\partial_1 \tau(u,v) \times \partial_2 \tau(u,v) = (v \cos u, \, v \sin u, \, -v), \qquad |\partial_1 \tau(u,v) \times \partial_2 \tau(u,v)| = \sqrt{2} \cdot v.$$

In addition, note that

$$G(\tau(u,v)) = G(v \cos u, \, v \sin u, \, v) = v^2 \cos^2 u + v^2 \sin^2 u = v^2.$$

Using the above, we can now evaluate the given surface integral:

$$\begin{aligned}
\iint_{\tau} G \, dA &= \iint_{(0,2\pi)\times(0,1)} (v^2 \cdot \sqrt{2} v) \, du dv \\
&= \sqrt{2} \int_0^{2\pi} du \int_0^1 v^3 \, dv \\
&= \sqrt{2} \cdot 2\pi \cdot \frac{1}{4} \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

(2) *(Intro to surface integrals)* One can also define an intermediate notion of surface integration of vector fields over *parametric surfaces*. More specifically:

Definition. Let $\sigma : U \to \mathbb{R}^3$ be a parametric surface, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *surface integral* of **F** over σ by

$$\iint_\sigma \mathbf{F} \cdot d\mathbf{A} = \iint_U \{\mathbf{F}(\sigma(u,v)) \cdot [\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v)]_{\sigma(u,v)}\} \, du dv.$$

(a) Consider the vector field **F** on $\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z) = \left(y,\ z^{5800} e^{x^{2000} + 46y^{1523}},\ x\right)_{(x,y,z)},$$

and let **P** be the parametric plane

$$\mathbf{P} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{P}(u,v) = (1,\ u,\ v).$$

Compute the surface integral of **F** over **P**.

(b) Consider the vector field **G** on $\mathbb{R}^3$ given by

$$\mathbf{G}(x,y,z) = (z,\ z,\ x^2 + y^2)_{(x,y,z)},$$

and let τ be the parametric torus

$$\tau : (0, 2\pi) \times (0,1) \to \mathbb{R}^3, \qquad \tau(u,v) = (v \cos u,\ v \sin u,\ v).$$

Compute the surface integral of **G** over τ.

(c) Consider the vector field **H** on $\mathbb{R}^3$ given by

$$\mathbf{H}(x,y,z) = (-x,\ -y,\ z)_{(x,y,z)},$$

and let **q** be the (regular) parametric surface

$$\mathbf{q} : (0,1) \times (0,1) \to \mathbb{R}^3, \qquad \mathbf{q}(u,v) = (u,\ v,\ u^2 + v^2).$$

Compute the surface integral of **H** over **q**.

(a) We begin by computing some preliminary quantities:

$$\begin{aligned} \partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v) &= (0,\ 1,\ 0) \times (0,\ 0,\ 1) \\ &= (1,\ 0,\ 0), \end{aligned}$$

$$
\begin{aligned}
\mathbf{F}(\mathbf{P}(u,v)) &= \mathbf{F}(1,u,v) \\
&= \left(u,\ v^{5800}e^{1+46u^{1523}},\ 1\right)_{\mathbf{P}(u,v)},
\end{aligned}
$$

for any $(u,v) \in (0,1) \times (u,1)$. The above then implies

$$
\begin{aligned}
\mathbf{F}(\mathbf{P}(u,v)) \cdot [\partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v)]_{\mathbf{P}(u,v)} &= \left(u,\ v^{5800}e^{1+46u^{1523}},\ 1\right) \cdot (1,\ 0,\ 0) \\
&= u.
\end{aligned}
$$

Thus, recalling our definition of parametric surface integral, we obtain

$$
\begin{aligned}
\iint\limits_{\mathbf{P}} \mathbf{F} \cdot d\mathbf{A} &= \iint\limits_{(0,1)\times(0,1)} \left\{ \mathbf{F}(\mathbf{P}(u,v)) \cdot [\partial_1 \mathbf{P}(u,v) \times \partial_2 \mathbf{P}(u,v)]_{\mathbf{P}(u,v)} \right\} du dv \\
&= \iint\limits_{(0,1)\times(0,1)} u \, du dv \\
&= \int_0^1 dv \int_0^1 u \, du \\
&= \frac{1}{2}.
\end{aligned}
$$

(b) First, we compute, for any $(u,v) \in (0, 2\pi) \times (0,1)$,

$$
\begin{aligned}
\partial_1 \tau(u,v) \times \partial_2 \tau(u,v) &= (-v \sin u,\ v \cos u,\ 0) \times (\cos u,\ \sin u,\ 1) \\
&= (v \cos u,\ v \sin u,\ -v) \\
\mathbf{G}(\tau(u,v)) &= (v,\ v,\ v^2 \cos^2 u + v^2 \sin^2 u)_{\tau(u,v)} \\
&= (v,\ v,\ v^2)_{\tau(u,v)}.
\end{aligned}
$$

Combining the above, we then obtain

$$
\begin{aligned}
\mathbf{G}(\tau(u,v)) \cdot [\partial_1 \tau(u,v) \times \partial_2 \tau(u,v)]_{\tau(u,v)} &= (v,\ v,\ v^2) \cdot (v \cos u,\ v \sin u,\ -v) \\
&= v^2 \cos u + v^2 \sin u - v^3.
\end{aligned}
$$

Thus, by our given definition of surface integrals,

$$
\begin{aligned}
\iint\limits_{\tau} \mathbf{G} \cdot d\mathbf{A} &= \iint\limits_{(0,2\pi)\times(0,1)} \left\{ \mathbf{G}(\tau(u,v)) \cdot [\partial_1 \tau(u,v) \times \partial_2 \tau(u,v)]_{\tau(u,v)} \right\} du dv \\
&= \int_0^1 \int_0^{2\pi} (v^2 \cos u + v^2 \sin u - v^3) \, du dv
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 (v^2 \sin u - v^2 \cos u - v^3 u)_{u=0}^{u=2\pi}\, dv \\
&= -2\pi \int_0^1 v^3\, dv \\
&= -\frac{\pi}{2}.
\end{aligned}
$$

(c) First, note that for any $(u, v) \times (0, 1) \times (0, 1)$, we have

$$
\begin{aligned}
\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v) &= (1,\ 0,\ 2u) \times (0,\ 1,\ 2v) \\
&= (-2u,\ -2v,\ 1), \\
\mathbf{H}(\mathbf{q}(u, v)) &= (-u,\ -v,\ u^2 + v^2)_{\mathbf{q}(u,v)}, \\
\mathbf{H}(\mathbf{q}(u, v)) \cdot [\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v)]_{\mathbf{q}(u,v)} &= (-u,\ -v,\ u^2 + v^2) \cdot (-2u,\ -2v,\ 1) \\
&= 3u^2 + 3v^2.
\end{aligned}
$$

Finally, using the above, we can evaluate the desired surface integral:

$$
\begin{aligned}
\iint_{\mathbf{q}} \mathbf{H} \cdot d\mathbf{A} &= \iint_{(0,1)\times(0,1)} \left\{ \mathbf{H}(\mathbf{q}(u, v)) \cdot [\partial_1 \mathbf{q}(u, v) \times \partial_2 \mathbf{q}(u, v)]_{\mathbf{q}(u,v)} \right\} du dv \\
&= \int_0^1 \int_0^1 (3u^2 + 3v^2)\, du dv \\
&= 2.
\end{aligned}
$$

(3) *(A Survey of Integration)* Let S denote the set

$$
S = \{(u, v, u^2 - v^2) \in \mathbb{R}^3 \mid (u, v) \in (0, 1) \times (0, 1)\}.
$$

(a) Show that S is a surface. In addition, give an injective parametrisation of S whose image is precisely all of S.

(b) Compute the surface integral over S of the real-valued function

$$
F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = xy.
$$

(The double integral you get from expanding the surface integral is not so pleasant; you will probably have to use the method of substitution twice to compute it.)

(c) Let us also assign to S the *upward-facing orientation*, i.e. the orientation in the *positive* *z-direction.* Then, compute the surface integral over S of the vector field

$$\mathbf{G}(x, y, z) = (xy^2,\ yx^2,\ 1)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$

(a) S is a surface, since it is the graph of the (smooth) function

$$f : (0, 1) \times (0, 1) \to \mathbb{R}, \qquad f(u, v) = u^2 - v^2.$$

Furthermore, an injective parametrisation of all of S is given by

$$\sigma : (0, 1) \times (0, 1) \to S, \qquad \sigma(u, v) = (u,\ v,\ u^2 - v^2).$$

(b) We begin by computing the partial derivatives of σ:

$$\partial_1 \sigma(u, v) = (1,\ 0,\ 2u), \qquad \partial_2 \sigma(u, v) = (0,\ 1,\ -2v).$$

Taking a cross product of the above yields

$$\begin{aligned} \partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v) &= (-2u,\ 2v,\ 1), \\ |\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| &= \sqrt{1 + 4u^2 + 4v^2}. \end{aligned}$$

Now, by part (a), we know that σ is an injective parametrisation of S whose image is all of S. Thus, we can use σ to compute our surface integral:

$$\iint_S F \, dA = \iint_\sigma F \, dA.$$

To calculate the above, we first note that

$$F(\sigma(u, v)) = F(u, v, u^2 - v^2) = uv.$$

Thus, our surface integral can now be expanded as

$$\iint_S F \, dA = \iint_{(0,1)\times(0,1)} \left(uv\sqrt{1 + 4u^2 + 4v^2} \right) du dv.$$

This can now be evaluated using Fubini's theorem and the method of subsitution:

$$\begin{aligned}
\iint\limits_S F\, dA &= \int_0^1 v \left[\int_0^1 u\sqrt{1+4u^2+4v^2}\, du\right] dv \\
&= \int_0^1 v \left[\frac{1}{12}(1+4u^2+4v^2)^{\frac{3}{2}}\right]_{u=0}^{u=1} dv \\
&= \frac{1}{12}\int_0^1 v\left[(5+4v^2)^{\frac{3}{2}} - (1+4v^2)^{\frac{3}{2}}\right] dv \\
&= \frac{1}{12}\cdot\frac{1}{20}\cdot\left[(5+4v^2)^{\frac{5}{2}} - (1+4v^2)^{\frac{5}{2}}\right]_{v=0}^{v=1} \\
&= \frac{1}{240}\left(9^{\frac{5}{2}} - 5^{\frac{5}{2}} - 5^{\frac{5}{2}} + 1^{\frac{5}{2}}\right) \\
&= \frac{61}{60} - \frac{5\sqrt{5}}{24}. \qquad \text{(Sorry ☺)}
\end{aligned}$$

(Even if you were not able to get the final number, the most important part is that you can correctly expand the surface integral into a double integral.)

(c) First, recall from part (b) that

$$\partial_1\sigma(u,v) \times \partial_2\sigma(u,v) = (-2u,\, 2v,\, 1),$$

hence it follows that σ generates the upward-facing orientation of S. Consequently, we can use the parametrisation σ to compute our surface integrals:

$$\iint\limits_S \mathbf{G}\cdot d\mathbf{A} = + \iint\limits_U \{\mathbf{G}(\sigma(u,v)) \cdot [\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)]_{\sigma(u,v)}\}\, du dv.$$

To calculate the above, we observe that

$$\mathbf{G}(\sigma(u,v)) = (uv^2,\, vu^2,\, 1)_{\sigma(u,v)},$$

and hence

$$\mathbf{G}(\sigma(u,v)) \cdot [\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)]_{\sigma(u,v)} = (uv^2,\, vu^2,\, 1)\cdot(-2u,\, 2v,\, 1) = 1.$$

Therefore, we conclude that

$$\iint\limits_S \mathbf{G}\cdot d\mathbf{A} = \iint\limits_{(0,1)\times(0,1)} 1\, du dv = 1.$$

(4) [**Marked**] Let S denote the following surface:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y^2 - z^2,\ 0 < y < 1,\ 0 < z < 1\}.$$

(a) Compute the surface integral over S of the function

$$G : \mathbb{R}^3 \to \mathbb{R}, \qquad G(x, y, z) = yz.$$

(b) Let us also assign to S the *orientation in direction of increasing* x*-value*. Then, compute the surface integral over S of the vector field $\mathbf{H}$ on $\mathbb{R}^3$ given by

$$\mathbf{H}(x, y, z) = (y, 0, z)_{(x,y,z)},$$

(a) The first step is to parametrise S appropriately. For this, we set $u = y$ and $v = z$:

$$\sigma : (0, 1) \times (0, 1) \to S, \qquad \sigma(u, v) = (u^2 - v^2,\ u,\ v).$$

Observe σ is injective, and its image is exactly S. [1 mark for correct parametrisation]

Furthermore, for any $(u, v) \in (0, 1) \times (0, 1)$, we compute

$$\begin{aligned} \partial_1\sigma(u, v) &= (2u,\ 1,\ 0), \\ \partial_2\sigma(u, v) &= (-2v,\ 0,\ 1), \\ \partial_1\sigma(u, v) \times \partial_2\sigma(u, v) &= (1, -2u, 2v), \\ |\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)| &= \sqrt{1 + 4u^2 + 4v^2}. \end{aligned}$$

Similarly, we have that

$$G(\sigma(u, v)) = uv.$$

We can now use σ and the above to compute the surface integral over S:

$$\begin{aligned} \iint_S G\, dA &= \iint_\sigma G\, dA \\ &= \iint_{(0,1)\times(0,1)} G(\rho(u, v))|\partial_1\rho(u, v) \times \partial_2\rho(u, v)|\, du dv \\ &= \int_0^1 v \int_0^1 u\sqrt{1 + 4u^2 + 4v^2}\, du dv. \end{aligned}$$

[1 mark for almost correct answer up to this point] From here, we directly compute

$$\begin{aligned}
\iint_S G\, dA &= \frac{1}{12}\int_0^1 v(1+4u^2+4v^2)^{\frac{3}{2}}|_{u=0}^{u=1} dv \\
&= \frac{1}{12}\int_0^1 [v(5+4v^2)^{\frac{3}{2}} - v(1+4v^2)^{\frac{3}{2}}] dv \\
&= \frac{1}{240}[(5+4v^2)^{\frac{5}{2}} - (1+4v^2)^{\frac{5}{2}}]_{v=0}^{v=1} \\
&= \frac{1}{240}[(9^{\frac{5}{2}} - 5^{\frac{5}{2}}) - (5^{\frac{5}{2}} - 1^{\frac{5}{2}})] \\
&= \frac{1}{240}(244 - 2\cdot 5^{\frac{5}{2}}).
\end{aligned}$$

[1 mark for somewhat correct integral]

(b) We can again use the parametrisation σ from (a). Observe that

$$[\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)]_{\sigma(u,v)} = (1, -2u, 2v)_{\sigma(u,v)},$$

which is in the normal direction generated by σ, points in the direction of increasing x (since the x-component is $+1$). Thus, σ generates our given orientation of S, and

$$\iint_S \mathbf{H}\cdot d\mathbf{A} = +\iint_{(0,1)\times(0,1)} \{\mathbf{H}(\sigma(u,v))\cdot[\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)]_{\sigma(u,v)}\}\, du dv.$$

[1 mark for correct observation of orientation]

The integral can now be computed directly. First, we have

$$\mathbf{H}(\sigma(u,v)) = (u,\ 0,\ v)_{\sigma(u,v)}.$$

Thus, combining all the above, we obtain that

$$\begin{aligned}
\iint_S \mathbf{H}\cdot d\mathbf{A} &= +\iint_{(0,1)\times(0,1)} [(u,0,v)\cdot(1,-2u,2v)]\, du dv \\
&= \int_0^1\int_0^1 (u+2v^2)\, du dv \\
&= \int_0^1 \left(\frac{1}{2}+2v^2\right) dv \\
&= \frac{1}{2}+\frac{2}{3}
\end{aligned}$$

$$= \frac{7}{6}.$$

[1 mark for an almost correct answer]

(5) [Tutorial]

(a) Consider the surface *(you may assume this is indeed a surface)*

$$\mathcal{P} = \{(u, v, u^4 + v) \in \mathbb{R}^3 \mid (u, v) \in (0, 1) \times (-1, 1)\}.$$

Compute the surface integral over $\mathcal{P}$ of the following function:

$$F : \mathbb{R}^3 \to \mathbb{R}, \qquad F(x, y, z) = 6x^5.$$

(b) Consider the sphere,

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and let $\mathbb{S}^2$ be given the "outward-facing" orientation. Compute the surface integral over $\mathbb{S}^2$ of the vector field $\mathbf{F}$ on $\mathbb{R}^3$ defined by the formula

$$\mathbf{F}(x, y, z) = (0, 0, z^3)_{(x,y,z)}.$$

(a) The first step is to appropriately parametrise $\mathcal{P}$. Observe that the map

$$\sigma : (0, 1) \times (-1, 1) \to \mathcal{P}, \qquad \sigma(u, v) = (u,\ v,\ u^4 + v)$$

is a parametrisation of $\mathcal{P}$. Moreover, note that σ is injective, and its image is all of $\mathcal{P}$. As a result, we have, from the definition of surface integrals,

$$\iint_{\mathcal{P}} F\, dA = \iint_{\sigma} F\, dA = \iint_{(0,1)\times(-1,1)} F(\sigma(u, v)) |\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)|\, du dv.$$

Next, the partial derivatives of σ satisfy

$$\partial_1 \sigma(u, v) = (1,\ 0,\ 4u^3), \qquad \partial_2 \sigma(u, v) = (0,\ 1,\ 1).$$

Thus, the required terms in the above integrand satisfy

$$\begin{aligned} |\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)| &= |(-4u^3,\ -1,\ 1)| = \sqrt{2+16u^6}, \\ F(\sigma(u,v)) &= 6u^5. \end{aligned}$$

Combining all the above, we can now compute the surface integral as

$$\begin{aligned} \iint_{\mathcal{P}} F\, dA &= \int_{-1}^{1}\int_0^1 6u^5\sqrt{2+16u^6}\, du dv \\ &= 2\int_0^1 6u^5\sqrt{2+16u^6}\, du \\ &= 2\cdot\frac{1}{16}\cdot\frac{2}{3}\cdot\left[(2+16u^6)^{\frac{3}{2}}\right]_{u=0}^{u=1} \\ &= \frac{13\sqrt{2}}{3}. \end{aligned}$$

(b) Recall (from lectures and the lecture notes) that the parametrisation of $\mathbb{S}^2$ given by

$$\rho : (0, 2\pi) \times (0, \pi) \to \mathbb{S}^2, \qquad \rho(u,v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v),$$

is injective, and that its image is "almost all" of $\mathbb{S}^2$ (the image excludes only two points and a semicircle). Moreover, from the usual computations, we have that

$$\partial_1\rho(u,v) \times \partial_2\rho(u,v) = -\sin v \cdot (\cos u \sin v,\ \sin u \sin v,\ \cos v) = -\sin v \cdot \rho(u,v).$$

In particular, the arrows

$$[\partial_1\rho(u,v) \times \partial_2\rho(u,v)]_{\rho(u,v)} = -\sin v \cdot \rho(u,v)_{\rho(u,v)},$$

which are normal to $\mathbb{S}^2$, point inward from $\mathbb{S}^2$. Thus, ρ generates the orientation opposite to our given orientation of $\mathbb{S}^2$, and hence we have that

$$\iint_{\mathbb{S}^2} \mathbf{F}\cdot d\mathbf{A} = -\iint_{(0,2\pi)\times(0,\pi)} \left\{\mathbf{F}(\rho(u,v)) \cdot [\partial_1\rho(u,v) \times \partial_2\rho(u,v)]_{\rho(u,v)}\right\} du dv.$$

Note the integrand satisfies

$$\mathbf{F}(\rho(u,v)) \cdot [\partial_1\rho(u,v) \times \partial_2\rho(u,v)]_{\rho(u,v)} = -\sin v(0,\ 0,\ \cos^3 v) \cdot (\cos u \sin v,\ \sin u \sin v,\ \cos v)$$

$$= -\sin v \cos^4 v.$$

As a result,

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{A} = \int_0^{2\pi} \int_0^{\pi} \sin v \cos^4 v \, dv du$$
$$= 2\pi \cdot \frac{1}{5} [-\cos^5 v]_{v=0}^{v=\pi}$$
$$= \frac{4\pi}{5}.$$

(6) *(A-levels, revisited)*

(a) Show that the surface area of a *sphere of radius* $r > 0$,

$$S_r = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\},$$

is equal to $4\pi r^2$.

(b) Show that the area of the side of a cone with base radius $r > 0$ and height $h > 0$,

$$C_{r,h} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid 0 < z < h,\ x^2 + y^2 = r^2 \left(1 - \frac{z}{h}\right)^2 \right\},$$

is equal to $\pi r \sqrt{r^2 + h^2}$.

(a) Similar to the case of a unit sphere, we see that

$$\rho_r : (0, 2\pi) \times (0, \pi) \to S_r, \qquad \rho_r(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$$

is an injective parametrisation of S_r, whose image is all of S_r except for two points and a semicircle. Moreover, a direct calculation (analogous to the one for $\mathbb{S}^2$) shows that

$$|\partial_1 \rho_r(u, v) \times \partial_2 \rho_r(u, v)| = |-r \sin v \cdot \rho_r(u, v)| = r^2 \sin v.$$

As a result, we obtain

$$\mathcal{A}(S_r) = \iint_{(0,2\pi)\times(0,\pi)} r^2 \sin v \, du dv = r^2 \int_0^{2\pi} du \int_0^{\pi} \sin v \, dv = 4\pi r^2.$$

(b) The main step is to parametrise $C_{r,h}$ correctly. For this, we can take

$$\sigma : (0, 2\pi) \times (0, h) \to C_{r,h}, \qquad \sigma(u, v) = (r(1 - vh^{-1}) \cos u, r(1 - vh^{-1}) \sin u, v).$$

In particular, σ is injective, and its image is all of $C_{r,h}$ except for a line. (Plot this out and see for yourself!) Moreover, direct computations yield

$$\begin{aligned}
\partial_1\sigma(u, v) &= (-r(1 - vh^{-1}) \sin u,\ r(1 - vh^{-1}) \cos u,\ 0),\\
\partial_2\sigma(u, v) &= (-rh^{-1} \cos u,\ -rh^{-1} \sin u,\ 1),\\
\partial_1\sigma(u, v) \times \partial_2\sigma(u, v) &= (r(1 - vh^{-1}) \cos u,\ r(1 - vh^{-1}) \sin u,\ r^2h^{-1}(1 - vh^{-1})),\\
|\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)| &= r\left(1 - \frac{v}{h}\right)\sqrt{1 + \left(\frac{r}{h}\right)^2}.
\end{aligned}$$

Combining the above, we conclude that the surface area is

$$\begin{aligned}
\mathcal{A}(C_{r,h}) &= r\sqrt{1 + \left(\frac{r}{h}\right)^2} \iint_{(0,2\pi)\times(0,h)} \left(1 - \frac{v}{h}\right) du dv\\
&= 2\pi r\sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h \left(1 - \frac{v}{h}\right) dv\\
&= 2\pi r\sqrt{1 + \left(\frac{r}{h}\right)^2} \cdot \frac{h}{2}\\
&= \pi r\sqrt{r^2 + h^2}.
\end{aligned}$$

(7) *(Reversal of orientations)* Let $S \subseteq \mathbb{R}^3$ be an oriented surface, and let $\sigma : U \to S$ be a parametrisation of S. Moreover, define the set

$$U_r = \{(v, u) \mid (u, v) \in U\}$$

and define the parametric surface

$$\sigma_r : U_r \to \mathbb{R}^3, \qquad \sigma_r(v, u) = \sigma(u, v).$$

In other words, σ_r is precisely σ but with the roles of u and v reversed.

(a) Show that for any $(u, v) \in U$,

$$\partial_1\sigma_r(v, u) \times \partial_2\sigma_r(v, u) = -[\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)].$$

(b) Show that σ_r is also a parametrisation of S, and that σ_r has the same image as σ.

(c) Use the formula from part (a) to conclude that if σ generates an orientation O of S, then σ_r generates the orientation opposite to O.

(a) We begin by relating the partial derivatives of σ and σ_r—for any $(v, u) \in U_r$,

$$\begin{aligned}\partial_1 \sigma_r(v, u) &= \partial_v[\sigma_r(v, u)] = \partial_v[\sigma(u, v)] = \partial_2 \sigma(u, v),\\ \partial_2 \sigma_r(v, u) &= \partial_u[\sigma_r(v, u)] = \partial_u[\sigma(u, v)] = \partial_1 \sigma(u, v).\end{aligned}$$

As a result, using that the cross product is antisymmetric, we conclude that

$$\begin{aligned}\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u) &= \partial_2 \sigma(u, v) \times \partial_1 \sigma(u, v)\\ &= -[\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)].\end{aligned}$$

(b) First, suppose $\mathbf{p}$ is in the image of σ, so that $\mathbf{p} = \sigma(u, v)$ for some $(u, v) \in U$. Then, by definition, $(v, u) \in U_r$ and $\sigma_r(v, u) = \sigma(u, v) = \mathbf{p}$, and it follows that $\mathbf{p}$ is also in the image of σ_r. Conversely, if $\mathbf{p}$ is in the image of σ_r, then $\mathbf{p} = \sigma_r(v, u)$ for some $(v, u) \in U_r$. This then implies $(u, v) \in U$ and $\sigma(u, v) = \sigma_r(v, u) = \mathbf{p}$, and hence $\mathbf{p}$ is also in the image of σ_r. From the above, we conclude that σ and σ_r have the same image.

In particular, the above implies that the image of σ_r lies within S. Moreover, using the formula obtained from part (a), we have, for any $(v, u) \in U_r$,

$$|\partial_1 \sigma_r(v, u) \times \partial_2 \sigma_r(v, u)| = |\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)| \neq 0,$$

since σ is regular by assumption. This implies that σ_r is also regular.

Combining the above, we conclude that σ_r is indeed a parametrisation of S.

(c) For any point $\mathbf{p} = \sigma(u, v) = \sigma_r(v, u)$ of S (where $(u, v) \in U$), we have that:

- The orientation generated by σ at $\mathbf{p}$ is given by

$$\mathbf{n}_\sigma(u, v) = + \left[\frac{\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)}{|\partial_1 \sigma(u, v) \times \partial_2 \sigma(u, v)|} \right]_{\sigma(u,v)}.$$

- Recalling the result from (a), the orientation selected by σ_r at $\mathbf{p}$ is given by

$$\begin{aligned}\mathbf{n}_{\sigma_r}(v,u) &= +\left[\frac{\partial_1\sigma_r(v,u)\times\partial_2\sigma_r(v,u)}{|\partial_1\sigma_r(v,u)\times\partial_2\sigma_r(v,u)|}\right]_{\sigma_r(v,u)} \\ &= -\left[\frac{\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)}{|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|}\right]_{\sigma(u,v)} \\ &= -\mathbf{n}_\sigma(u,v).\end{aligned}$$

In particular, the above shows that σ_r generates the opposite unit normals as σ, and hence σ_r generates the orientation opposite to that of σ.

(8) *(The paradox of Gabriel's horn)* Consider the surface of revolution

$$G = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, y^2+z^2 = \frac{1}{x^2},\ x>1\right\},$$

which is sometimes nicknamed *Gabriel's horn*. *(Before proceeding, you should search for "Gabriel's horn" on Google Images to see an illustration of* G*.)*

(a) Show that G has infinite surface area.

(b) Show that the interior of G,

$$I = \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, y^2+z^2 \leq \frac{1}{x^2},\ x>1\right\},$$

has finite volume.

In other words, *you can fill up the inside of the "horn" with a finite amount of paint, but you cannot paint the "horn" itself using a finite amount of paint*!

(a) To compute the surface area, we first parametrise G appropriately:

$$\sigma : (1,\infty)\times(0,2\pi)\to G, \qquad \sigma(u,v) = (u,\ u^{-1}\cos v,\ u^{-1}\sin v).$$

Note in particular that σ is injective, and its image is all of G except for a curve. (The reasoning here is analogous to that for Question (4).)

Next, we do some computations involving σ:

$$\partial_1\sigma(u,v) = (1,\ -u^{-2}\cos v,\ -u^{-2}\sin v),$$

$$\begin{aligned}
\partial_2\sigma(u,v) &= (0,\ -u^{-1}\sin v,\ u^{-1}\cos v),\\
\partial_1\sigma(u,v)\times\partial_2\sigma(u,v) &= (-u^{-3},\ -u^{-1}\cos v,\ -u^{-1}\sin v),\\
|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)| &= u^{-1}\sqrt{1+u^{-4}},
\end{aligned}$$

Combining the above with the definition of surface area, we conclude that

$$\begin{aligned}
A(G) &= \iint\limits_{(1,\infty)\times(0,2\pi)} |\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)|\, du dv\\
&= \int_0^{2\pi} dv \int_1^\infty \frac{1}{u}\sqrt{1+\frac{1}{u^4}}\, du.
\end{aligned}$$

Since $1+u^{-4}\geq 1$ for all $u\in\mathbb{R}$, it follows that

$$A(G)\geq 2\pi\int_1^\infty \frac{1}{u}\, du = \lim_{u\nearrow\infty} \ln u - \ln 1 = +\infty.$$

Thus, we conclude that $A(G)$ is indeed infinite.

(b) Recall the volume of I is

$$V(I) = \iiint\limits_I 1\, dxdydz.$$

The easiest way to describe I in a way that is convenient for integration is to do a change of variables and write y and z in terms of polar coordinates:

$$x = x, \qquad y = r\cos\theta, \qquad z = r\sin\theta.$$

In particular, I can be described in these new coordinates as

$$I = \{(x,r,\theta)\in\mathbb{R}^3 \mid x>1,\ 0\leq r\leq x^{-1},\ 0\leq\theta\leq 2\pi\}.$$

Note that the Jacobian with respect to this change of variables is

$$J = \det\frac{\partial(x,y,z)}{\partial(x,r,\theta)} = \det\begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\theta & -r\sin\theta\\ 0 & \sin\theta & r\cos\theta\end{bmatrix} = r.$$

Thus, by the change of variables formula and Fubini's theorem, we have that

$$\begin{aligned} V(I) &= \int_0^{2\pi} \int_1^{\infty} \int_0^{x^{-1}} J \, dr dx d\theta \\ &= \int_0^{2\pi} d\theta \int_1^{\infty} \int_0^{x^{-1}} r \, dr dx \\ &= 2\pi \cdot \frac{1}{2} \int_1^{\infty} \frac{1}{x^2} \, dx \\ &= \pi. \end{aligned}$$

Thus, the volume of I is indeed finite.