MTH5113 (Winter 2022): Problem Sheet 6
Solutions

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(1) *(Warm-up)* For each of the sets C and points $\mathbf{p} \in C$ given below:

(i) Show that C is a curve.

(ii) Sketch C, and indicate the point $\mathbf{p}$ on C.

(iii) Give a parametrisation of C that passes through $\mathbf{p}$.

(a) *Hyperbola:*
$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = -1\}, \qquad \mathbf{p} = (0, -1).$$

(b) *Cubic:*
$$C = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^3 = y - 3\}, \qquad \mathbf{p} = (0, -5).$$

(c) *Ellipse:*
$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 6x + 4y^2 - 8y = 3\}, \qquad \mathbf{p} = (-3, -1).$$

(a) (i) Note that C can be written as the following level set:

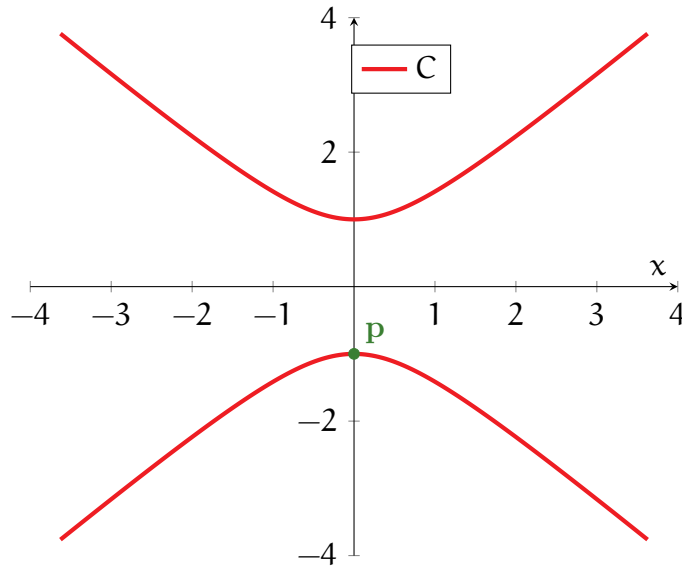$$C = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = -1\},$$

where f is the function
$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = x^2 - y^2.$$

Observe that the gradient of f satisfies

$$\nabla f(x,y) = (2x, -2y)_{(x,y)},$$

which vanishes only when $(x,y) = (0,0) \notin C$. Thus, the "level set theorem" (from either the lectures or the lecture notes) implies that C is indeed a curve.

(ii) A sketch is given below, with C in red and $\mathbf{p}$ in green:

(iii) One approach is to set x to be the parameter t. Note that y must satisfy

$$y^2 = 1 + x^2, \qquad y = \pm\sqrt{1 + x^2} = \pm\sqrt{1 + t^2}.$$

Since this parametrisation must pass through $\mathbf{p} = (0, -1)$, it follows that we must choose the negative branch in the above. As a result, one possible parametrisation of C is

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(t, -\sqrt{1 + t^2}\right).$$

(Note in particular γ maps out the bottom half of C, and that $\gamma(0) = (0, -1)$.)

If you like hyperbolic functions, then a slicker parametrisation of the bottom half of C is

$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (\sinh t, -\cosh t).$$

(Note in particular that $\lambda(0) = (0, -1)$.)

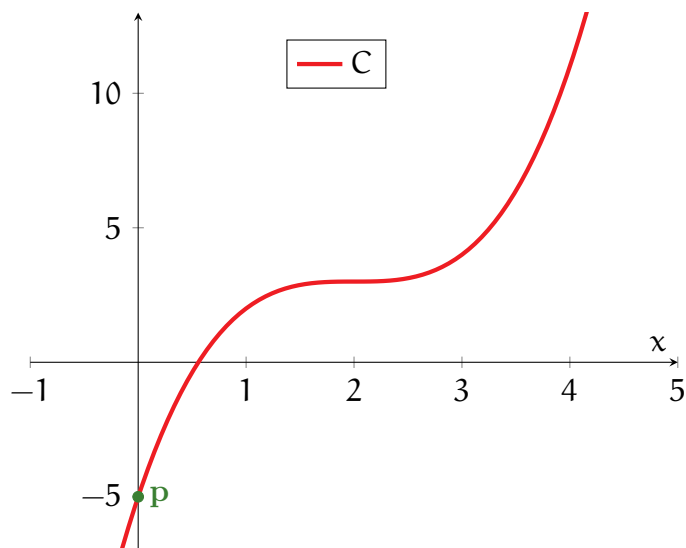**(b)** (i) Here, we need simply note that C is the graph of the function

$$c : \mathbb{R} \to \mathbb{R}, \qquad c(x) = (x - 2)^3 + 3,$$

that is,

$$C = \{(t, c(t)) \mid t \in \mathbb{R}\}.$$

It follows (from theorems in lectures or the lecture notes) that C is indeed a curve.

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(ii) A sketch is given below, with C in red and $\mathbf{p}$ in green:

(iii) Since C is the graph of c from part (i), we have a natural way to parametrise all of C:

$$\gamma_c : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma_c(t) = (t, c(t)) = (t, (t-2)^3 + 3).$$

Note in particular that $\gamma_c(0) = (0, -5)$.

(c) This one is a bit trickier. First, we can write the defining equation for C as

$$16 = (x^2 + 6x + 9) + 4(y^2 - 2y + 4) = (x+3)^2 + 4(y-1)^2.$$

In other words, we can express

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x+3)^2 + 4(y-1)^2 = 16\},$$

from which we can conclude that C is an ellipse centred at $(-3, 1)$.

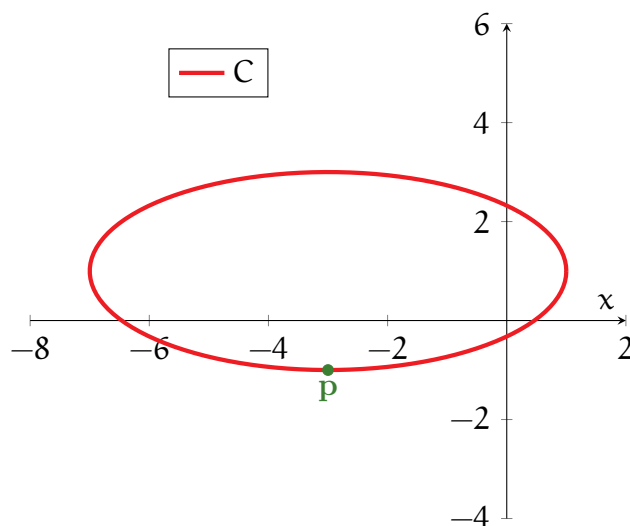(i) To show C is a curve, we first note that it is a level set of the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = (x+3)^2 + 4(y-1)^2.$$

Note that the gradient of g satisfies

$$\nabla g(x, y) = (2(x+3), \, 8(y-1))_{(x,y)},$$

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which only vanishes when $(x, y) = (-3, 1)$. Since $(-3, 1) \notin C$ (this can be checked using the definition of C), it follows that C is indeed a curve.

(ii) A sketch is given below, with C in red and $\mathbf{p}$ in green:

(iii) The nicest parametrisation is obtained by relating C to a circle. Setting

$$\tilde{x} = x + 3, \qquad \tilde{y} = 2(y - 1),$$

then the defining equation for C is becomes

$$\tilde{x}^2 + \tilde{y}^2 = 4^2.$$

Since the above describes a circle about the origin of radius 4, this can be parametrised as $\tilde{x} = 4 \cos t$ and $\tilde{y} = 4 \sin t$. Putting the above in terms of x and y yields

$$x = -3 + 4 \cos t, \qquad y = 1 + 2 \sin t.$$

The above implies that the following is a valid parametrisation of (all of) C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (-3 + 4 \cos t, \ 1 + 2 \sin t).$$

(Note, in particular, that $(-3, -1) = \gamma(-\frac{\pi}{2})$.)

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Alternatively, one can set $x = t$ and solve for y. This yields

$$\lambda : (-7, 1) \to \mathbb{R}^2, \qquad \lambda(t) = \left(t, 1 - \sqrt{4 - \frac{1}{4}(t+3)^2} \right).$$

In this case, $(-3, -1) = \lambda(-3)$.

(2) *(Fun with plotting)* The following are exercises involving sketching parametric surfaces. Do make use of computer programs or webpages (see the links in the *Additional Resources* section on the *QMPlus page*) to help you with your sketches.

(a) *Sphere:* Consider the following parametric surface:

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \sin v, \sin u \sin v, \cos v).$$

(i) Sketch the paths obtained from σ by holding v constant, with values

$$v_0 = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi.$$

(ii) Sketch the paths obtained from σ by holding u constant, with values

$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of σ.

(b) *Hyperboloid:* Consider the following parametric surface:

$$\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{h}(u, v) = (\cos u \cosh v, \sin u \cosh v, \sinh v).$$

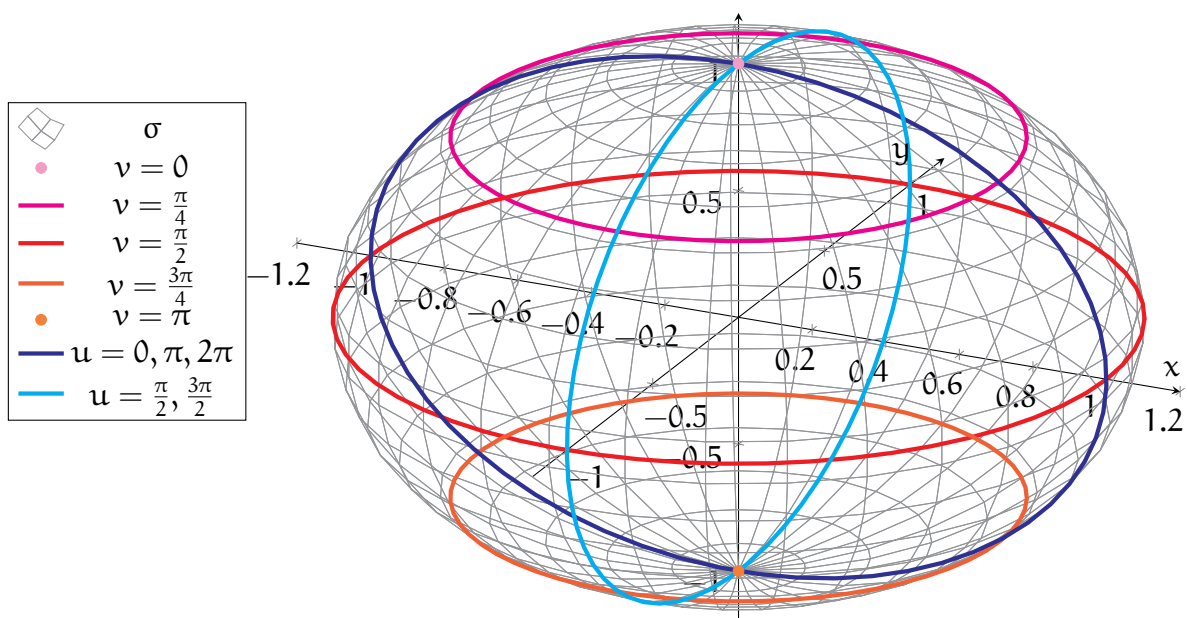(i) Sketch the paths obtained from $\mathbf{h}$ by holding v constant, with values

$$v_0 = -2, -1, 0, 1, 2.$$

(ii) Sketch the paths obtained from $\mathbf{h}$ by holding u constant, with values

$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of $\mathbf{h}$.

(a) A sketch of σ is given below; moreover, the paths obtained by holding v constant are drawn in red, while the paths obtained by holding u constant are drawn in blue:

(b) A sketch of **h** is given below; moreover, the paths obtained by holding v constant are drawn in red, while the paths obtained by holding u constant are drawn in blue:
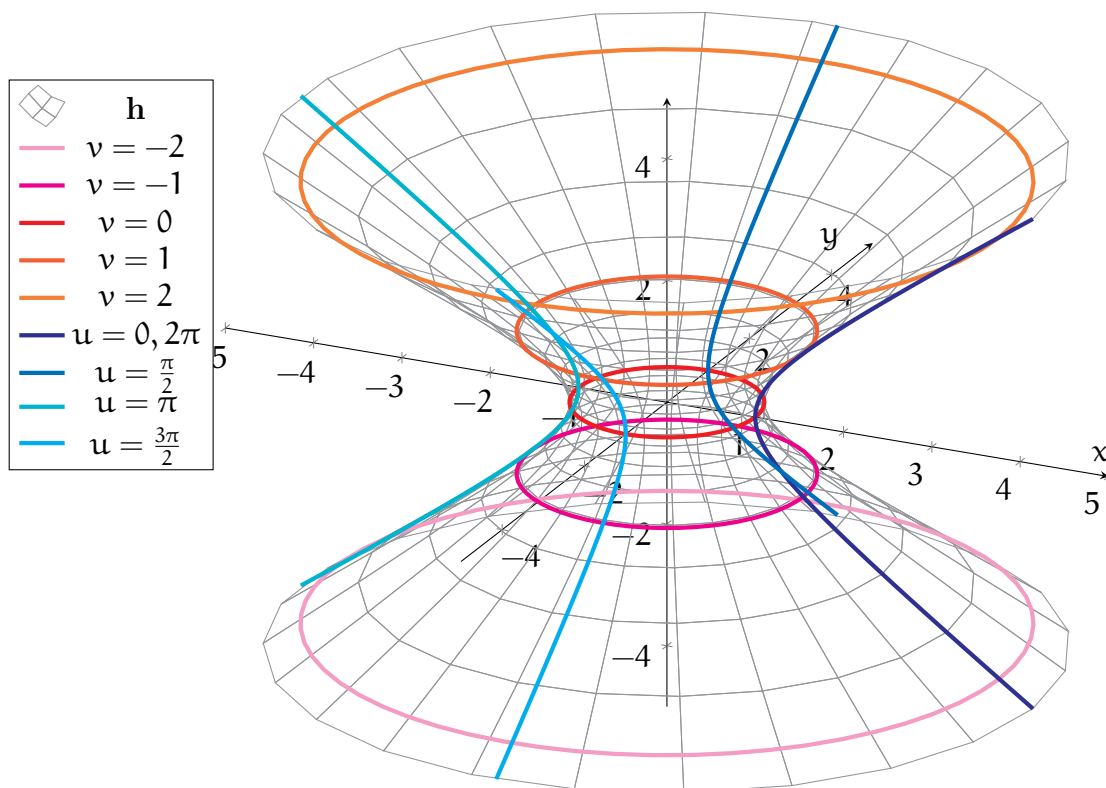
(3) *(Warm-up)* For each of the following parametric surfaces σ and parameters (u_0, v_0), compute the tangent plane to σ at (u_0, v_0):

(a) σ is the parametric *cylinder*,

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u,\ \sin u,\ v),$$

and $(u_0, v_0) = (\frac{\pi}{2}, -1)$.

(b) σ is the parametric *one-sheeted hyperboloid,*

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

and $(u_0, v_0) = (\pi, 1)$.

(a) We begin by differentiating σ:

$$\partial_1 \sigma(u, v) = (-\sin u,\ \cos u,\ 0), \qquad \partial_2 \sigma(u, v) = (0,\ 0,\ 1).$$

Evaluating the above at $(\frac{\pi}{2}, -1)$ yields

$$\sigma\left(\frac{\pi}{2}, -1\right) = (0, 1, -1), \qquad \partial_1 \sigma\left(\frac{\pi}{2}, -1\right) = (-1, 0, 0), \qquad \partial_2 \sigma\left(\frac{\pi}{2}, -1\right) = (0, 0, 1).$$

Thus, by definition, we conclude that

$$T_\sigma\left(-\frac{\pi}{2}, -1\right) = \left\{ a \cdot \partial_1 \sigma\left(\frac{\pi}{2}, -1\right)_{\sigma(\frac{\pi}{2}, -1)} + b \cdot \partial_2 \sigma\left(\frac{\pi}{2}, -1\right)_{\sigma(\frac{\pi}{2}, -1)} \,\middle|\, a, b \in \mathbb{R} \right\}$$
$$= \left\{ a \cdot (-1, 0, 0)_{(0,1,-1)} + b \cdot (0, 0, 1)_{(0,1,-1)} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

(b) Taking partial derivatives of σ yields

$$\partial_1 \sigma(u, v) = (-\sin u \cosh v,\ \cos u \cosh v,\ 0),$$
$$\partial_2 \sigma(u, v) = (\cos u \sinh v,\ \sin u \sinh v,\ \cosh v).$$

Evaluating the above at $(\pi, 1)$, we obtain

$$\sigma(\pi, 1) = (-\cosh 1,\ 0,\ \sinh 1),$$
$$\partial_1 \sigma(\pi, 1) = (0,\ -\cosh 1,\ 0),$$

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(4) *(Introduction to curve integrals)* One can also define an intermediate notion of curve integration of vector fields over *parametric curves*. More specifically:

Definition. Let $\gamma : (a, b) \to \mathbb{R}^n$ be a parametric curve, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *curve integral* of $\mathbf{F}$ over γ by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b [\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt.$$

For each of the following γ and $\mathbf{F}$, compute the curve integral of $\mathbf{F}$ over γ:

(a) γ is the regular parametric curve

$$\gamma : (0, 1) \to \mathbb{R}^3, \qquad \gamma(t) = (t,\, -t,\, 2t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x, y, z)_{(x,y,z)}.$$

(b) γ is the regular parametric curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (\cos t,\, \sin t \cos t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (x^2, 0)_{(x,y)}.$$

(a) The first step is to compute, for each $t \in (0, 1)$,

$$\gamma'(t) = (1, -1, 2), \qquad \mathbf{F}(\gamma(t)) = \mathbf{F}(t, -t, 2t) = (t, -t, 2t)_{(t,-t,2t)}.$$

Thus, by the above definition, we obtain

$$\begin{aligned}
\int_\gamma \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} \right] dt \\
&= \int_0^1 \left[(t, -t, 2t)_{(t,-t,2t)} \cdot (1, -1, 2)_{(t,-t,2t)} \right] dt \\
&= \int_0^1 \left[(t, -t, 2t) \cdot (1, -1, 2) \right] dt \\
&= \int_0^1 6t \, dt \\
&= 3.
\end{aligned}$$

(b) Again, we begin by computing the necessary quantities:

$$\gamma'(t) = (-\sin t, \cos^2 t - \sin^2 t), \qquad \mathbf{F}(\gamma(t)) = (\cos^2 t, 0)_{(\cos t, \sin t \cos t)}.$$

As a result, we see that

$$\begin{aligned}
\int_\gamma \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} \right] dt \\
&= \int_0^{2\pi} \left[(\cos^2 t, 0) \cdot (-\sin t, \cos^2 t - \sin^2 t) \right] dt \\
&= -\int_0^{2\pi} \cos^2 t \sin t \, dt \\
&= \frac{1}{3} [\cos^3 t]_{t=0}^{t=2\pi} \\
&= 0.
\end{aligned}$$

(5) [Marked] Consider the *clockwise-oriented* (shifted) *circle*,

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x + 2)^2 + (y - 3)^2 = 9\},$$

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and consider the vector field $\mathbf{F}$ on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (y, -x)_{(x,y)}.$$

(a) Give an *injective* parametrisation γ of C such that *the image of γ differs from C by only a finite number of points. Which orientation of C does γ generate?*

(b) Compute the (curve) integral of $\mathbf{F}$ over the curve C.

(a) Note that C is the circle of radius 3 centred at $(-2, 3)$. Thus, one simple parametrisation of C is given by shifting the usual circular parametrisation by $(-2, 3)$ and rescaling:

$$\gamma : (0, 2\pi) \to C, \qquad \gamma(t) = (-2 + 3\cos t, 3 + 3\sin t).$$

Note γ is injective, and its image is all of C except for a single point $(1, 3)$ (corresponding to $t = 0$ and $t = 2\pi$). [1 mark for correct parametrisation] Moreover, note γ generates the anticlockwise orientation of C. [1 mark for observing the correct orientation for γ]

(b) We can integrate using γ from part (a). First, note that

$$\gamma'(t) = (-3\sin t, \, 3\cos t), \qquad \mathbf{F}(\gamma(t)) = (3 + 3\sin t, \, 2 - 3\cos t)_{\gamma(t)}.$$

[1 mark if mostly correct to this point.] Thus, taking a dot product yields

$$\begin{aligned}
\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} &= (3 + 3\sin t, \, 2 - 3\cos t) \cdot (-3\sin t, \, 3\cos t) \\
&= -9\sin t - 9\sin^2 t + 6\cos t - 9\cos^2 t \\
&= -9 + 6\cos t - 9\sin t.
\end{aligned}$$

Since γ generates the orientation opposite to that of C, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_0^{2\pi} \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)} \right] dt = \int_0^{2\pi} (9 - 6\cos t + 9\sin t).$$

[1 mark if mostly correct to this point.] All that remains is a direct computation:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = [9t - 6\sin t - 9\cos t]_{t=0}^{t=2\pi} = 18\pi.$$

[1 mark for mostly correct answer]

(6) [Tutorial] For each of the following oriented curves C and vector fields $\mathbf{F}$:

(i) Give an *injective* parametrisation γ of C such that *the image of γ differs from C by only a finite number of points. Which orientation does γ generate?*

(ii) Compute the (curve) integral of $\mathbf{F}$ over the curve C.

(a) C is the *anticlockwise-oriented ellipse*,

$$C = \{(x, y) \in \mathbb{R}^2 \mid 3x^2 + 2y^2 = 6\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (y, -x)_{(x,y)}.$$

(b) C is the *downward-oriented* (with decreasing z-value) *helical segment*,

$$C = \{(\cos t, \sin t, t) \mid t \in (0, 2\pi)\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-y, x, 1)_{(x,y,z)}.$$

(a) (i) For this, we can borrow the formula from Question (5b) of Problem Sheet 4:

$$\gamma(t) = \left(\sqrt{2} \cdot \cos t, \ \sqrt{3} \cdot \sin t\right).$$

We can make it injective by restricting its domain so that it only traverses C once:

$$\gamma : (0, 2\pi) \to C, \qquad \gamma(t) = \left(\sqrt{2} \cdot \cos t, \ \sqrt{3} \cdot \sin t\right).$$

(In particular, the image of image is all of C except for its rightmost point $(\sqrt{2}, 0)$.) Furthermore, notice that γ generates the anticlockwise orientation of C.

(ii) We can now integrate $\mathbf{F}$ using γ from part (i):

$$\int_C \mathbf{F} \cdot d\mathbf{s} = + \int_0^{2\pi} \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}\right] dt.$$

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By direct computations, we see that

$$\gamma'(t) = \left(-\sqrt{2} \cdot \sin t, \ \sqrt{3} \cdot \cos t\right),$$

$$\mathbf{F}(\gamma(t)) = \left(\sqrt{3} \cdot \sin t, \ -\sqrt{2} \cdot \cos t\right)_{\left(\sqrt{2} \cdot \cos t, \sqrt{3} \cdot \sin t\right)}.$$

Thus, combining the above yields

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \left[\left(\sqrt{3} \cdot \sin t, \ -\sqrt{2} \cdot \cos t\right) \cdot \left(-\sqrt{2} \cdot \sin t, \ \sqrt{3} \cdot \cos t\right)\right] dt$$

$$= -\sqrt{6} \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt$$

$$= -2\sqrt{6}\,\pi.$$

(b) (i) We can use the apparent parametrisation of C,

$$\lambda : (0, \ 2\pi) \to C, \qquad \lambda(t) = (\cos t, \ \sin t, \ t),$$

which is injective and whose image is all of C. Notice, however, that λ has the upward orientation, which is opposite to the given orientation of C.

(ii) The integral can now be computed using λ from part (i):

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -\int_0^{2\pi} \left[\mathbf{F}(\lambda(t)) \cdot \lambda'(t)_{\lambda(t)}\right] dt$$

$$= -\int_0^{2\pi} [(-\sin t, \ \cos t, \ 1) \cdot (-\sin t, \ \cos t, \ 1)] \, dt$$

$$= -\int_0^{2\pi} (\sin^2 t + \cos^2 t + 1) \, dt$$

$$= -4\pi.$$

(7) *(Exploring curvature)* Let $\gamma : I \to \mathbb{R}^n$ be any regular parametric curve. We then define the *curvature* of γ at $t \in I$ by the formula

$$\kappa_\gamma(t) = \frac{1}{|\gamma'(t)|} \left| \left(\frac{\gamma'}{|\gamma'|}\right)'(t) \right|.$$

(This can be viewed as the "change in the direction of γ per unit length"; see the *2019*

version of the *MTH5113 lecture notes* for additional discussions of curvature.)

(a) Let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$, and let ℓ be the parametric line,

$$\ell : \mathbb{R} \to \mathbb{R}^n, \qquad \ell(t) = \mathbf{p} + t\mathbf{v}.$$

Compute the curvature of ℓ at every $t \in \mathbb{R}$.

(b) Let $R > 0$, and let γ_R be the parametric circle of radius R:

$$\gamma_R : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma_R(t) = (R \cos t, \, R \sin t).$$

Compute the curvature of γ_R at every $t \in \mathbb{R}$.

(c) Show that *curvature is independent of parametrisation.* More specificially, show that if γ is a reparametrisation of $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$, with corresponding change of variables $\phi : I \to \tilde{I}$ (in particular, $\gamma(t) = \tilde{\gamma}(\phi(t))$ for all $t \in I$), then

$$\kappa_\gamma(t) = \kappa_{\tilde{\gamma}}(\phi(t)), \qquad t \in I.$$

(a) To compute the curvature of ℓ, we first calculate

$$\ell'(t) = \mathbf{v}, \qquad |\ell'(t)| = |\mathbf{v}|, \qquad \frac{\ell'(t)}{|\ell'(t)|} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Since all three of the above are constant, we see from our given definition that

$$\kappa_\ell(t) = \frac{1}{|\ell'(t)|} \left| \left(\frac{\ell'}{|\ell'|} \right)'(t) \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d}{dt} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \right| = 0, \qquad t \in \mathbb{R}.$$

(b) First, some direct computations yield

$$\gamma_R'(t) = (-R \sin t, \, R \cos t), \qquad |\gamma_R'(t)| = R, \qquad \frac{\gamma_R'(t)}{|\gamma_R'(t)|} = (-\sin t, \cos t).$$

Applying our given definition to the above then yields, for any $t \in \mathbb{R}$,

$$\kappa_{\gamma_R}(t) = \frac{1}{R} \left| \frac{d}{dt}(-\sin t, \cos t) \right| = \frac{1}{R} |(-\cos t, -\sin t)| = \frac{1}{R}.$$

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(c) Recalling the relation between γ' and $\tilde{\gamma}'$ (see lectures or the lecture notes), we have that

$$
\begin{aligned}
\kappa_\gamma(t) &= \frac{1}{|\gamma'(t)|}\left|\left(\frac{\gamma'}{|\gamma'|}\right)'(t)\right| \\
&= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left[\frac{\phi'\cdot\tilde{\gamma}'(\phi)}{|\phi'|\cdot|\tilde{\gamma}'(\phi)|}\right]'(t)\right| \\
&= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left[\frac{\tilde{\gamma}'(\phi)}{|\tilde{\gamma}'(\phi)|}\right]'(t)\right|,
\end{aligned}
$$

where $\tilde{\gamma}'(\phi)$ denotes the composition of $\tilde{\gamma}'$ with ϕ, and where the last step follows from the observation that $|\phi'(t)|^{-1}\phi'(t)$ has constant value ± 1.

Furthermore, applying the chain rule to the outer derivative in the above, we see that

$$
\begin{aligned}
\kappa_\gamma(t) &= \frac{1}{|\phi'(t)||\tilde{\gamma}'(\phi(t))|}\left|\left(\frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}\right)'(\phi(t))\cdot\phi'(t)\right| \\
&= \frac{1}{|\tilde{\gamma}'(\phi(t))|}\left|\left(\frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}\right)'(\phi(t))\right| \\
&= \kappa_{\tilde{\gamma}}(\phi(t)),
\end{aligned}
$$

as desired. (In the first step, we applied the chain rule to the function $|\tilde{\gamma}'|^{-1}\cdot\tilde{\gamma}'$.)

(8) *(Polar curves)* Let $h:\mathbb{R}\to\mathbb{R}$ be a smooth positive periodic function, with period 2π:

$$
h(\theta) > 0, \qquad h(\theta + 2\pi) = h(\theta), \qquad x \in \mathbb{R}.
$$

Let the *polar curve* P be the set of all points in $\mathbb{R}^2$ satisfying the relation

$$
r = h(\theta)
$$

in polar coordinates. (Here, you can assume that P is indeed a curve.)

  (a) The unit circle $\mathcal{C} = \{(x,y)\in\mathbb{R}\mid x^2+y^2=1\}$ is a polar curve. What is h here?

  (b) Give a injective parametrisation of P whose image is all of P except for a single point.

  (c) Derive a formula for the arc length of P.

(a) Since the expression $x^2 + y^2$ is simply r^2 in polar coordinates, it follows that $\mathcal{C}$ is the polar curve $r = 1$, that is, with $h(\theta) = 1$ for all $\theta \in \mathbb{R}$.

(b) The most straightforward method is to set the parameter t to be the polar angle θ. (This is analogous to the polar parametrisation of the unit circle). Since

$$x = r\cos\theta = h(\theta)\cos\theta, \qquad y = r\sin\theta = h(\theta)\sin\theta,$$

and since the above starts repeating itself after θ has increased by 2π, then the following is an injective parametrisation of P whose image misses only a single point of P:

$$\gamma : (0, 2\pi) \to P, \qquad \gamma(t) = (h(t)\cos t,\, h(t)\sin t).$$

(In particular, γ misses the point $\gamma(0) = (h(0),\, 0)$.

(c) We use the parametrisation in part (b). Note that

$$|\gamma'(t)| = |(h'(t)\cos t - h(t)\sin t,\, h'(t)\sin t + h(t)\cos t)| = \sqrt{[h'(t)]^2 + [h(t)]^2}.$$

As a result, the arc length of P is

$$L(P) = L(\gamma) = \int_0^{2\pi} \sqrt{[h(t)]^2 + [h'(t)]^2}\, dt.$$

(See also Question (5c) of Problem Sheet 5.)

(9) *(Conic sections)* Let N denote the following cone:

$$N = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}.$$

In addition, let $P \subseteq \mathbb{R}^3$ denote an arbitrary plane that does not pass through the origin. More specifically, P is a set of the form

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$$

where $a, b, c, d \in \mathbb{R}$ satisfy $(a, b, c) \neq (0, 0, 0)$ and $d \neq 0$. A set of the form $N \cap P$ (i.e. the intersection of the cone N and the plane P) is called a *conic section*.

    (a) Use the theorem in Question (9) of Problem Sheet 4 to show that any conic section $N \cap P$ is indeed a curve. *(Hint: You will have to be resourceful to do this. The first step is to express $N \cap P$ as an appropriate level set.)*

    (b) Find examples of such planes P such that the conic section $N \cap P$ is:

(i) A circle.

(ii) An ellipse.

(iii) A parabola.

(iv) A hyperbola.

Check your answers graphically on a computer!

(a) We begin by observing that $N \cap P$ can be expressed as a level set

$$N \cap P = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ g(x, y, z) = d\}, \tag{1}$$

where the functions f and g are given by

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = x^2 + y^2 - z^2,$$
$$g : \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = ax + by + cz.$$

Taking the gradients of f and g yields, for any $(x, y, z) \in \mathbb{R}^3$,

$$\nabla f(x, y, z) = (2x,\ 2y,\ -2z)_{(x,y,z)}, \qquad \nabla g(x, y, z) = (a,\ b,\ c)_{(x,y,z)}.$$

Recall that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ vanishes if and only if the two gradients point in the same or in the opposite directions. Since $(a, b, c) \neq (0, 0, 0)$ by assumption, it follows that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ vanishes if and only if there is some $\lambda \in \mathbb{R}$ such that

$$(x, y, -z) = \lambda(a, b, c).$$

We now let L denote the set

$$L = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, -z) = \lambda(a, b, c) \text{ for some } \lambda \in \mathbb{R}\}.$$

(Note that L is a line through the origin.) We claim that *no point of L lies in $N \cap P$*. To show this, we suppose $(x, y, z) \in L$—so that $(x, y, -z) = \lambda(a, b, c)$ for some $\lambda \in \mathbb{R}$—and we assume (x, y, z) does lie in $N \cap P$. Then, (1) implies that

$$x^2 + y^2 - z^2 = 0, \qquad x^2 + y^2 - z^2 = \lambda(ax + by + cz) = \lambda d. \tag{2}$$

Since $d \neq 0$ by assumption, (2) yields $\lambda = 0$. However, this implies $(x, y, z) = (0, 0, 0)$, which by assumption does not lie in $N \cap P$, resulting in a contradiction. This proves the claim.

Combining all the above, we deduce that $\nabla f(x, y, z) \times \nabla g(x, y, z)$ does not vanish at any point of $N \cap P$. Therefore, applying the theorem in Question (9) of Problem Sheet 4 to $N \cap P$ in (1), we conclude that it is indeed a curve.

(b) (i) One example of a circle is to take $N \cap P$, with
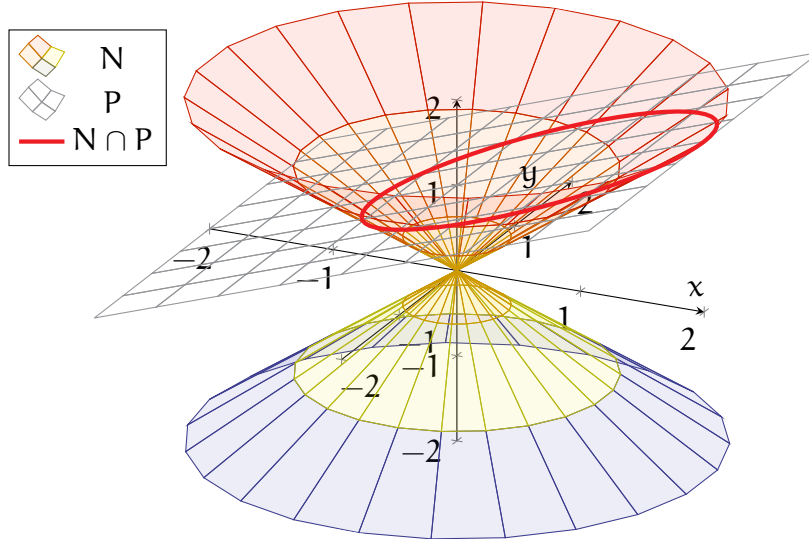
$$P = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\}.$$

(In general, any P with $(a, b) = (0, 0)$ suffices.) This is illustrated below:
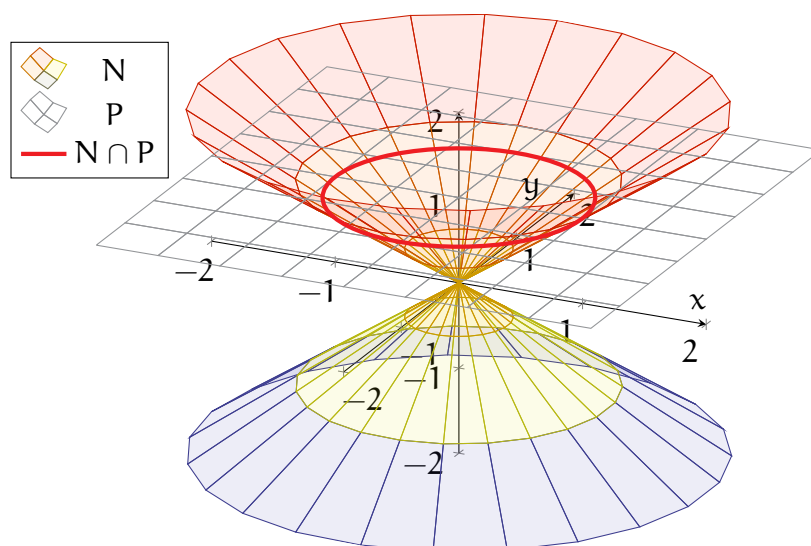
(ii) One example of a (non-circular) ellipse is to take $N \cap P$, with

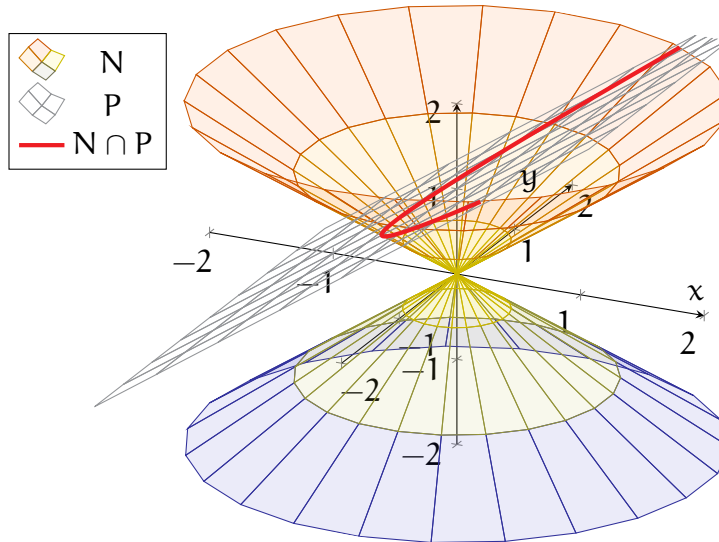$$P = \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = \frac{1}{2}x + 1 \right\}.$$

(In general, any P with $a^2 + b^2 < c^2$ suffices.) This is illustrated below:

(iii) One example of a parabola is to take $N \cap P$, with

$$P = \left\{ (x, y, z) \in \mathbb{R}^3 \mid z = x + 1 \right\}.$$

(In general, any P with $a^2 + b^2 = c^2$ suffices.) This is illustrated below:

(iv) One example of a hyperbola is to take $N \cap P$, with

$$P = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x = 1 \right\}.$$

(In general, any P with $a^2 + b^2 > c^2$ suffices.) This is illustrated below: