MTH5113 (Winter 2022): Problem Sheet 5
Solutions

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(1) *(Warm-up)* Given a curve C, along with a pair of parametrisations γ_1 and γ_2 of C:

(i) Compute, for each parameter t, the unit tangent vectors

$$\frac{1}{|\gamma_1'(t)|} \cdot \gamma_1'(t)_{\gamma_1(t)}, \qquad \frac{1}{|\gamma_2'(t)|} \cdot \gamma_2'(t)_{\gamma_2(t)}.$$

(ii) Determine whether γ_1 and γ_2 generate the same orientation or opposite orientations of C. Give a brief justification of your answer.

(a) *Circle:* $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and

$$\begin{aligned} \gamma_1 &: \mathbb{R} \to C, & \gamma_1(t) &= (\cos t, \sin t), \\ \gamma_2 &: \mathbb{R} \to C, & \gamma_2(t) &= (-\sin t, -\cos t). \end{aligned}$$

(b) *Line:* $C = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 2,\ x + z = 1\}$, and

$$\begin{aligned} \gamma_1 &: \mathbb{R} \to C, & \gamma_1(t) &= (t,\ 2 - t,\ 1 - t), \\ \gamma_2 &: \mathbb{R} \to C, & \gamma_2(t) &= (2 - t,\ t,\ t - 1). \end{aligned}$$

(a) (i) Differentiating γ_1 and γ_2 yields:

$$\begin{aligned} \gamma_1'(t) &= (-\sin t, \cos t), & |\gamma_1'(t)| &= 1, \\ \gamma_2'(t) &= (-\cos t, \sin t), & |\gamma_2'(t)| &= 1. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \frac{1}{|\gamma_1'(t)|} \cdot \gamma_1'(t)_{\gamma_1(t)} &= (-\sin t, \cos t)_{(\cos t, \sin t)}, \\ \frac{1}{|\gamma_2'(t)|} \cdot \gamma_2'(t)_{\gamma_2(t)} &= (-\cos t, \sin t)_{(-\sin t, \cos t)}. \end{aligned}$$

(ii) γ_1, γ_2 generate opposite orientations of C. To see this, we compare the tangent vectors from part (i) at a common point and check that they point in opposite directions.

For instance, at $t = 0$ and $t = -\frac{\pi}{2}$, respectively, we have

$$\frac{1}{|\gamma_1'(0)|} \cdot \gamma_1'(0)_{\gamma_1(0)} = (0, 1)_{(1,0)},$$
$$\frac{1}{|\gamma_2'(-\frac{\pi}{2})|} \cdot \gamma_2' \left(-\frac{\pi}{2}\right)_{\gamma_2(-\frac{\pi}{2})} = (0, -1)_{(1,0)}.$$

In particular, the above shows that γ_1 and γ_2 are travelling in the anticlockwise and clockwise directions, respectively, at the point $(1, 0)$ (and hence at every point).

(b) (i) First, we differentiate γ_1 and γ_2 to obtain

$$\gamma_1'(t) = (1, -1, -1), \qquad |\gamma_1'(t)| = \sqrt{3},$$
$$\gamma_2'(t) = (-1, 1, 1), \qquad |\gamma_2'(t)| = \sqrt{3}.$$

From this, we conclude that

$$\frac{1}{|\gamma_1'(t)|} \cdot \gamma_1'(t)_{\gamma_1(t)} = \frac{1}{\sqrt{3}} \cdot (1, -1, -1)_{(t,2-t,1-t)},$$
$$\frac{1}{|\gamma_2'(t)|} \cdot \gamma_2'(t)_{\gamma_2(t)} = \frac{1}{\sqrt{3}} \cdot (-1, 1, 1)_{(2-t,t,t-1)}.$$

(ii) Clearly, γ_1 and γ_2 generate opposite orientations of C, since they are always pointing in constant but opposite directions.

(2) *(Warm-up)* Find both the unit tangents and the unit normals to each of the following curves C at the given point $\mathbf{p}$.

(a) $C = \{(t^3, t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$ and $\mathbf{p} = (-8, -2)$.

(b) $C = \{(x, y) \in \mathbb{R}^2 \mid x^4 + 2y^2 = 3\}$ and $\mathbf{p} = (-1, 1)$.

(a) First, note that the following is a parametrisation of C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (t^3, t).$$

Observe that $t = -2$ corresponds to the point $\mathbf{p}$:

$$\gamma(-2) = (-8, -2) = \mathbf{p}.$$

We also note that the derivative of γ satisfies

$$\gamma'(t) = (3t^2, 1), \qquad \gamma'(-2) = (12, 1).$$

Thus, the unit tangents to C at $\mathbf{p}$ are given by

$$\mathbf{t}_\mathbf{p}^\pm = \pm\frac{1}{|\gamma'(-2)|} \cdot \gamma'(-2)_{\gamma(-2)} = \pm\frac{1}{\sqrt{145}} \cdot (12, 1)_{(-8,-2)}.$$

The unit normals to C at $\mathbf{p}$ can then be obtained by rotating $\mathbf{t}_\mathbf{p}^\pm$:

$$\mathbf{n}_\mathbf{p}^\pm = \pm\frac{1}{\sqrt{145}} \cdot (-1, 12)_{(-8,-2)}.$$

(b) (This can be solved in the same way as in (a), through a parametrisation of C. However, here we show an alternate method that can be used when C is a level set.)

Observe that C can be expressed as a level set,

$$C = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = 3\}, \qquad h(x, y) = x^4 + 2y^2.$$

In particular, we compute

$$\nabla h(x, y) = (4x^3, 4y)_{(x,y)}, \qquad \nabla h(-1, 1) = (-4, 4)_{(-1,1)}, \qquad |\nabla h(-1, 1)| = 4\sqrt{2}.$$

As a result, the unit normals to C at $\mathbf{p} = (-1, 1)$ are

$$\mathbf{n}_\mathbf{p}^\pm = \pm\frac{1}{|\nabla h(-1, 1)|} \cdot \nabla h(-1, 1) = \pm\frac{1}{\sqrt{2}} \cdot (-1, 1)_{(-1,1)}.$$

The unit tangents can then be obtained from the unit normals via a $90°$ rotation:

$$\mathbf{t}_\mathbf{p}^\pm = \pm\frac{1}{\sqrt{2}} \cdot (1, 1)_{(-1,1)}.$$

(3) *(Warm-up)* Consider the following regular parametric curve:

$$\mathbf{b} : (0, 1) \to \mathbb{R}^2, \qquad \mathbf{b}(t) = \left(t,\ \frac{2}{3}\, t^{\frac{3}{2}}\right).$$

(a) Compute the arc length of $\mathbf{b}$.

(b) Compute the curve integral of the function F over $\mathbf{b}$, where

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad F(x, y) = 1 + x.$$

(c) Compute the curve integral of the function G over $\mathbf{b}$, where

$$G : \mathbb{R} \times (0, \infty) \to \mathbb{R}, \qquad G(x, y) = \frac{2}{3} + \frac{y}{\sqrt{x}}.$$

(a) First, we compute the derivative of $\mathbf{b}$ and its norm for any $t \in (0, 1)$:

$$\mathbf{b}'(t) = \left(1, \sqrt{t}\right), \qquad |\mathbf{b}'(t)| = \sqrt{1+t}.$$

By the definition of arc length,

$$L(\mathbf{b}) = \int_0^1 |\mathbf{b}'(t)| \, dt = \int_0^1 \sqrt{1+t} \, dt.$$

The above can then be integrated directly:

$$L(\mathbf{b}) = \frac{2}{3} (1+t)^{\frac{3}{2}} \Big|_{t=0}^{t=1} = \frac{2}{3} \left(2\sqrt{2} - 1\right).$$

(b) The first step is to notice that

$$F(\mathbf{b}(t)) = F\left(t, t^{\frac{3}{2}}\right) = 1 + t, \qquad t \in (0, 1).$$

Combining the computations from part (a) with the definition of curve integrals, we obtain

$$\int_{\mathbf{b}} F \, ds = \int_0^1 F(\mathbf{b}(t)) \, |\mathbf{b}'(t)| \, dt = \int_0^1 (1+t)\sqrt{1+t} \, dt = \int_0^1 (1+t)^{\frac{3}{2}} \, dt.$$

Integrating the above yields the solution:

$$\int_{\mathbf{b}} F \, ds = \frac{2}{5} (1+t)^{\frac{5}{2}} \Big|_{t=0}^{t=1} = \frac{2}{5} \left(4\sqrt{2} - 1\right).$$

(c) Similar to part (b), we first compute, for any $t \in (0,1)$,

$$G(\mathbf{b}(t)) = G\left(t, \frac{2}{3}t^{\frac{3}{2}}\right) = \frac{2}{3} + \frac{\frac{2}{3}t^{\frac{3}{2}}}{\sqrt{t}} = \frac{2}{3}(1+t).$$

Recalling the computations we have done in part (b), we obtain

$$\int_{\mathbf{b}} G\, ds = \frac{2}{3}\int_0^1 (1+t)\sqrt{1+t}\, dt = \frac{4}{15}\left(4\sqrt{2}-1\right).$$

(4) [**Marked**] Consider the following ellipse,

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 = 4\},$$

and consider the following function,

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad F(x,y) = \sqrt{1+3y^2}.$$

(a) Give an *injective* parametrisation γ of C such that *the image of* γ *differs from* C *by only a finite number of points*. Be sure to specify the domain of γ! (Hint: Recall all the possible solutions to Question (5) of Problem Sheet 4.)

(b) Compute the curve integral of F over C.

(a) One needs extra care to produce a parametrisation whose image covers almost all of C. The most straightforward way to do is to set $a = x$ and $b = 2y$, so that C is given by the equation $a^2 + b^2 = 4$. In particular, all of C can then be described parametrically as

$$a = 2\cos t, \qquad b = 2\sin t.$$

Switching to back to the xy-plane, we obtain the parametric relations

$$x = a = 2\cos t, \qquad y = \frac{b}{2} = \sin t.$$

Finally, to ensure our parametrisation is injective while still covering almost all of C, we restrict our values of t to one period of sin and cos, for instance, $t \in (0, 2\pi)$. Putting all this together, we see that one valid parametrisation of C is given by

$$\gamma : (0, 2\pi) \to C, \qquad \gamma(t) = (2\cos t, \sin t).$$

[1 mark for a correct domain] [1 mark for correct parametric formula]

Note that γ is injective, and its image is all of C except for a single point $(2, 0) \in C$. (See the alternate solution to Question (5b) of Problem Sheet 4 for a similar derivation.)

(b) To integrate F over C, we use the parametrisation γ from (a), which satisfies all the conditions in the definition of curve integrals. From this, we obtain

$$\int_C F\, ds = \int_\gamma F\, ds = \int_0^{2\pi} F(\gamma(t))|\gamma'(t)|\, dt.$$

Some quick computations yield, for any $t \in (0, 2\pi)$,

$$|\gamma'(t)| = |(-2\sin t, \cos t)| = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{1 + 3\sin^2 t},$$
$$F(\gamma(t)) = F(2\cos t, \sin t) = \sqrt{1 + 3\sin^2 t}.$$

[1 mark if mostly correct to this point.] Combining all the above, we conclude that

$$\int_C F\, ds = \int_0^{2\pi} (1 + 3\sin^2 t)\, dt = 2\pi + 3\int_0^{2\pi} \sin^2 t\, dt.$$

[1 mark if mostly correct to this point.]

While you could Google the last integral :(, you could also more cleverly do it on your own. For instance, recalling the trigonometric identity $\cos(2t) = 1 - 2\sin^2 t$, we then obtain

$$\int_0^{2\pi} \sin^2 t = \frac{1}{2}\int_0^{2\pi} [1 - \cos(2t)]\, dt = \frac{1}{2}\left[t - \frac{1}{2}\sin(2t)\right]_{t=0}^{t=2\pi} = \pi.$$

Finally, combining everything, we obtain

$$\int_C F\, ds = 2\pi + 3\int_0^{2\pi} \sin^2 t\, dt = 2\pi + 3\pi = 5\pi.$$

[1 mark for mostly correct answer.]

(5) **[Tutorial]** Let us derive some (possibly) familiar formulas!

(a) Let $\mathcal{C}$ denote the unit circle,

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Show that at each $\mathbf{p} \in \mathcal{C}$, the unit normals to $\mathcal{C}$ at $\mathbf{p}$ are precisely $\pm\mathbf{p}_\mathbf{p}$.

(b) Let G_f denote the graph of a function $f : (a, b) \to \mathbb{R}$:

$$G_f = \{(x, y) \in \mathbb{R}^2 \mid y = f(x),\ a < x < b\}.$$

Find a formula for the arc length of G_f.

(c) Let $\rho : (c, d) \to \mathbb{R}$, and consider the following *polar parametric curve*:

$$\lambda_\rho : (c, d) \to \mathbb{R}^2, \qquad \lambda_\rho(\theta) = (\rho(\theta)\cos\theta,\ \rho(\theta)\sin\theta).$$

Find a formula for the arc length of λ_ρ.

(a) Recall $\mathcal{C}$ is a level set, i.e.

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid s(x, y) = 1\}, \qquad s(x, y) = x^2 + y^2.$$

Moreover, the gradient of s satisfies, at any $(x, y) \in C$,

$$\nabla s(x, y) = (2x, 2y)_{(x,y)}, \qquad |\nabla s(x, y)| = 2\sqrt{x^2 + y^2} = 2.$$

(In the last step, we noted that $x^2 + y^2 = 1$, since $(x, y) \in C$.)

As a result, the unit normals to C at any $\mathbf{p} = (x, y) \in C$ are

$$\pm\frac{1}{|\nabla s(x, y)|} \cdot \nabla s(x, y) = \pm(x, y)_{(x,y)} = \pm\mathbf{p}_\mathbf{p}.$$

(b) We begin by finding an appropriate parametrisation of G_f:

$$\gamma_f : (a, b) \to \mathbb{R}^2, \qquad \gamma_f(t) = (t,\ f(t)).$$

Note γ_f is an injective parametrisation of G_f, and its image is precisely G_f.

Thus, by the definition of arc length, we obtain a familiar definition from calculus:

$$L(G_f) = L(\gamma_f) = \int_a^b |\gamma_f'(t)|\, dt = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt.$$

(c) First, we compute that

$$\begin{aligned}
\lambda_\rho'(\theta) &= (\rho'(\theta)\,\cos\theta - \rho(\theta)\,\sin\theta,\ \rho'(\theta)\,\sin\theta + \rho(\theta)\,\cos\theta), \\
|\lambda_\rho'(\theta)| &= \sqrt{[\rho'(\theta)\,\cos\theta - \rho(\theta)\,\sin\theta]^2 + [\rho'(\theta)\,\sin\theta + \rho(\theta)\,\cos\theta]^2} \\
&= \sqrt{[\rho'(\theta)]^2\,\cos^2\theta + [\rho(\theta)]^2\,\sin^2\theta + [\rho'(\theta)]^2\,\sin^2\theta + [\rho(\theta)]^2\,\cos^2\theta} \\
&= \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}.
\end{aligned}$$

As a result, by definition, we obtain another oft-mentioned calculus formula:

$$L(\lambda_\rho) = \int_c^d \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}\,d\theta.$$

(6) *(Issues with parametrisations)* Let H denote the curve

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid x = \cosh z,\ y = \sinh z\},$$

and let γ be the parametric curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\cosh t,\ \sinh t,\ t).$$

(For this problem, you may assume that you already know H is a curve.)

(a) What statements must you prove in order to show that γ is a parametrisation of H, according to the definition given in this module?

(b) Show that γ is indeed a parametrisation of H.

(c) Oh no, Mr. Error (from question (6) of *Problem Sheet 4*) is back to his erroneous ways! He decides to describe the points of H using the parametric curve

$$\zeta : \mathbb{R} \to H, \qquad \zeta(t) = (\cosh t^2,\ \sinh t^2,\ t^2).$$

He computes (correctly) that

$$\zeta(0) = (1, 0, 0) \in H, \qquad \zeta'(0) = (0, 0, 0).$$

He then (incorrectly) concludes that $T_{(1,0,0)}H$ contains only one element,

$$T_{(1,0,0)}H = T_\zeta(0) = \{s \cdot \zeta'(0)_{\zeta(0)} \mid s \in \mathbb{R}\} = \{(0, 0, 0)_{(1,0,0)}\},$$

hence it is a 0-dimensional space! What error did Mr. Error make this time?

(a) According to the definition, in order to show that γ is a parametrisation of H, we must prove that γ is regular, and that the image of γ lies in H.

(b) To show that γ is regular, we compute

$$\gamma'(t) = (\sinh t,\ \cosh t,\ 1).$$

In particular, the above never vanishes (the z-component is always 1). Thus, γ is regular.

Furthermore, for any $t \in \mathbb{R}$, the point

$$(x, y, z) = \gamma(t) = (\cosh t,\ \sinh t,\ t)$$

indeed satisfies both $x = \cosh z$ and $y = \sinh z$. Thus, the image of γ lies in H.

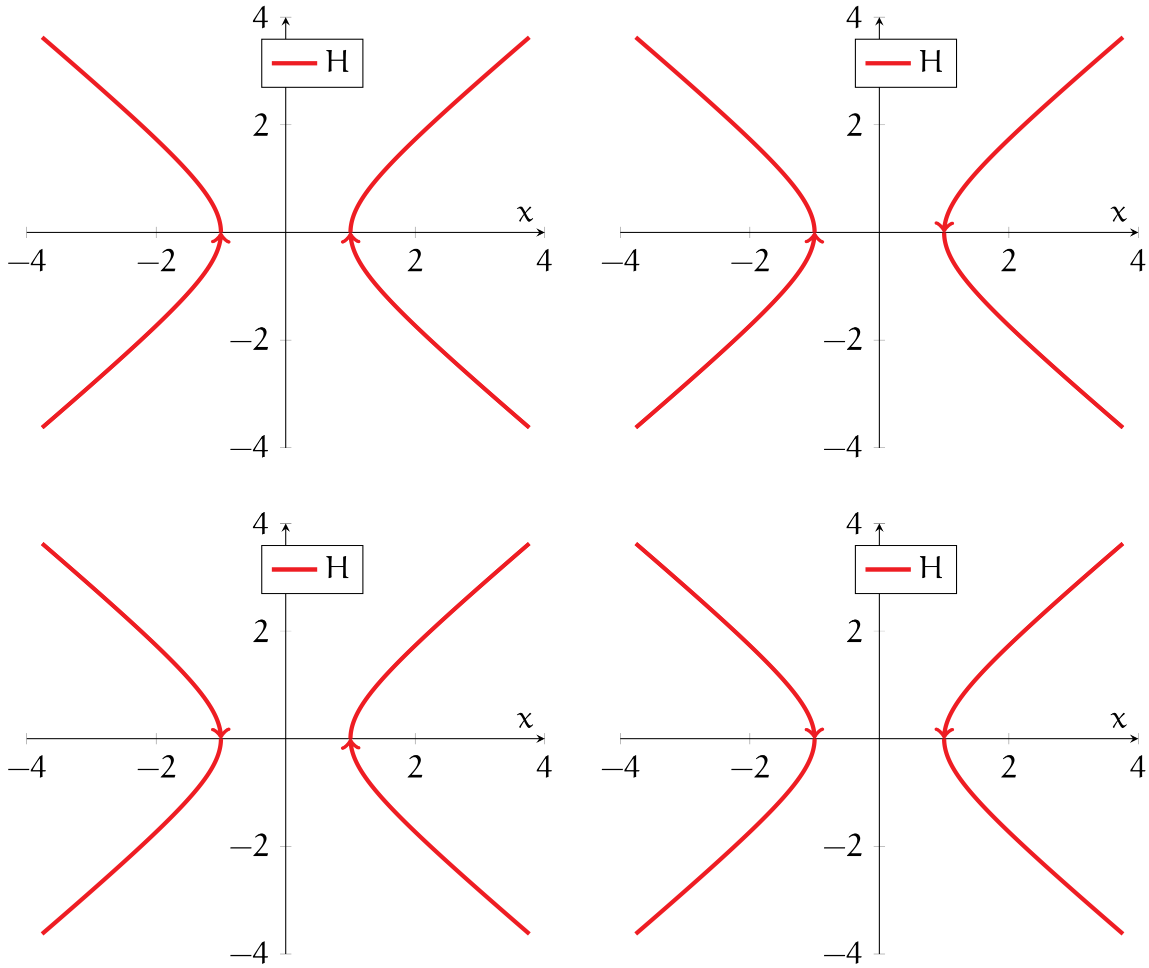
(c) Mr. Error's error is that ζ fails to be regular (since $\zeta'(0) = (0, 0, 0)$), and hence ζ, by definition, fails to be a parametrisation of H. In particular, at $t = 0$ (where ζ' vanishes), we see that ζ fails to capture the directions along H. (This shows that to capture the geometry of a curve, the requirement that a parametrisation is regular is essential.)

(7) *(Orient my hyperbola!)* Let $\mathcal{H}$ denote the *hyperbola*,

$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}.$$

Describe all the possible orientations of $\mathcal{H}$. How many such orientations are there?

There are four possible orientations of $\mathcal{H}$!



Recall that an orientation of $\mathcal{H}$ is a choice of continuously varying unit tangents at each point of $\mathcal{H}$. Moreover, note that $\mathcal{H}$ consists of two disconnected paths (see the plots below). As a result, one can freely choose a direction of travel on both the left and the right components of $\mathcal{H}$, independently of the choice on the other component.

Thus, the four orientations of $\mathcal{H}$ are described as follows:

- Upward on the left component, upward on the right component.
- Upward on the left component, downward on the right component.
- Downward on the left component, upward on the right component.
- Downward on the left component, downward on the right component.

These orientations are illustrated in the above illustrations.

(8) *(Arc length parametrisations)* Let (a, b) be a finite open interval, and let $\gamma : (a, b) \to \mathbb{R}^n$ be a regular parametric curve. We can then define the change of variables

$$s = \phi(t) = \int_a^t |\gamma'(\tau)| \, d\tau.$$

Note that s *represents the total length travelled by* γ *up to parameter* t.

(a) Show that the following holds for any $t \in (a, b)$:

$$\frac{ds}{dt} = |\gamma'(t)|.$$

The reparametrisation λ of γ defined as

$$\lambda : (0, L(\gamma)) \to \mathbb{R}^n, \qquad \lambda(s) = \gamma(t)$$

is called the *arc length reparametrisation*, since its parameter is itself the distance travelled.

(b) Let $R > 0$, and let γ be the regular parametric curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (R \cos t, \ R \sin t).$$

Find the arc length reparametrisation λ of this γ. What is the domain of λ?

(a) This is an immediate consequence of the (second) fundamental theorem of calculus:

$$\frac{ds}{dt} = \frac{d}{dt} \int_a^t |\gamma'(\tau)| \, d\tau = |\gamma'(t)|.$$

(b) For this specific γ, the arc length parameter is given by

$$s = \int_0^t |\gamma'(\tau)| \, d\tau = R \int_0^t d\tau = Rt.$$

As a result, the arc length reparametrisation of γ is defined by the formula

$$\lambda(s) = \gamma(t) = \gamma\left(\frac{s}{R}\right) = \left(R \cos \frac{s}{R}, \ R \sin \frac{s}{R}\right).$$

Moreover, since t lies in the range $(0, 2\pi)$, then $s = Rt$ must lie in the range $(0, 2\pi R)$. As a result, the precise arc length reparametrisation of γ is given by

$$\lambda : (0,\, 2\pi R) \to \mathbb{R}^2, \qquad \lambda(s) = \left(R\cos\frac{s}{R},\, R\sin\frac{s}{R} \right).$$

(9) *(Parental advisory, implicit content)* *(Not examinable)* A special case of the *implicit function theorem* for functions of two variables can be stated as follows:

> **Theorem.** *(Implicit Function Theorem)* Let $U \subseteq \mathbb{R}^2$ be open and connected, let $f : U \to \mathbb{R}$ be smooth, and let C denote the level set
>
> $$C = \{(x, y) \in U \mid f(x, y) = c\}, \qquad c \in \mathbb{R}.$$
>
> Suppose, in addition, that $(x, y) \in C$ and that $\partial_2 f(x, y) \neq 0$. Then, there exists some open set $V \subseteq \mathbb{R}^2$, with $(x, y) \in V$, such that $C \cap V$ is the graph of a function, i.e.,
>
> $$C \cap V = \{(x, h(x)) \mid x \in I\},$$
>
> where I is an open interval, and where $h : I \to \mathbb{R}$ is smooth.

In addition, an analogous theorem holds with the roles of x and y interchanged.

- **(a)** How does the above implicit function theorem relate to the process of implicit differentiation that you learned in calculus? (An informal description will suffice here.)
- **(b)** How does the above implicit function theorem relate to the proof of Theorem 3.26 in the lecture notes? Again, an informal description will suffice here.

(a) In implicit differentiation, one considers all the points (x, y) satisfying some relation:

$$C = \{(x, y) \in U \mid f(x, y) = c\}.$$

At each point $(x, y) \in C$, one then implicitly differentiates with respect to x:

$$\partial_1 f(x, y) + \partial_2 f(x, y)\, \frac{dy}{dx} = 0.$$

However, for this to make any sense, one must (implicitly) make a crucial assumption: *that y is actually a function of x on C*, or at least on a part of C.

The implicit function theorem guarantees that as long as f is sufficiently nice (i.e. $\partial_2 f$ is nonzero), then y indeed is (at least locally) a function of x on C.

(b) Consider a level set

$$C = \{(x, y) \in U \mid f(x, y) = c\}.$$

Suppose, as in Theorem 3.26 of the lecture notes, that $(x, y) \in C$ and $\nabla f(x, y) \neq 0$. Then, it follows that either $\partial_1 f(x, y) \neq 0$ or $\partial_2 f(x, y) \neq 0$.

If $\partial_2 f(x, y) \neq 0$, then the implicit function theorem implies an open set V, containing (x, y), such that $C \cap V$ is the graph of a function $h : I \to \mathbb{R}$. From this, we see that $C \cap V$ can be described using an injective parametrisation of γ:

$$\gamma : I \to C, \qquad \gamma(t) = (t, h(t)).$$

(One can in fact show that this γ satisfies all the conditions for the formal definition of a curve, however we omit the details here.)

Similarly, if $\partial_1 f(x, y) \neq 0$ instead, then using the analogue of the implicit function theorem with the roles x and y interchanged, we see that the points of C near (x, y) take the form

$$C \cap V = \{(g(y), y) \mid y \in J\}, \qquad g : J \to \mathbb{R}.$$

Thus, a portion of C near (x, y) can be injectively parametrised in the form

$$\lambda : I \to C, \qquad \gamma(t) = (g(t), t).$$

(Again, λ can be shown to satisfy all the conditions for the formal definition of a curve.)