MTH5113 (Winter 2022): Problem Sheet 4
Solutions

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(1) *(Warm-up)*

(a) Consider the following regular parametric curve:

$$\mathbf{h} : \mathbb{R} \to \mathbb{R}^3, \qquad \mathbf{h}(t) = (t, \cos t, \sin t).$$

(i) Find the tangent line to $\mathbf{h}$ at $t = \frac{3\pi}{4}$.

(ii) Sketch $\mathbf{h}$, the tangent vector $\mathbf{h}'(\frac{3\pi}{4})_{\mathbf{h}(\frac{3\pi}{4})}$, and the tangent line from (i).

(b) Consider the following regular parametric curve:

$$\mathbf{k} : (0, \infty) \to \mathbb{R}^2, \qquad \mathbf{k}(t) = (t \cos t, t \sin t).$$

(i) Find the tangent line to $\mathbf{k}$ at $t = \pi$.

(ii) Sketch $\mathbf{k}$, the tangent vector $\mathbf{k}'(\pi)_{\mathbf{k}(\pi)}$, and the tangent line from (i).

(a) Differentiating $\mathbf{h}$ yields

$$\mathbf{h}'(t) = (1, -\sin t, \cos t).$$

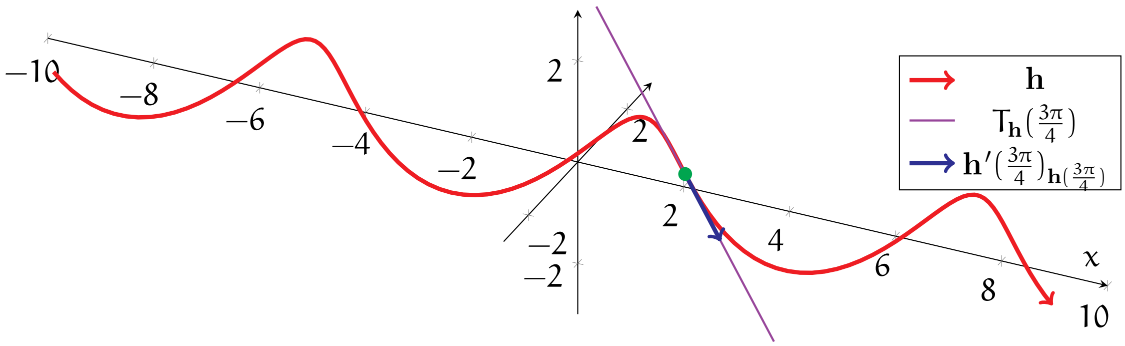
Evaluating $\mathbf{h}$ and $\mathbf{h}'$ at $t = \frac{3\pi}{4}$ then yields

$$\mathbf{h}\left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{4}, \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\right) = \left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$
$$\mathbf{h}'\left(\frac{\pi}{4}\right) = \left(1, -\sin\frac{3\pi}{4}, \cos\frac{3\pi}{4}\right) = \left(1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Thus, by definition, the tangent line to $\mathbf{h}$ at $t = \frac{3\pi}{4}$ is

$$T_{\mathbf{h}}\left(\frac{3\pi}{4}\right) = \left\{ s \cdot \left(1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)_{\left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \,\middle|\, s \in \mathbb{R} \right\}.$$

A sketch of $\mathbf{h}$ (in red), $\mathbf{h}'(\frac{3\pi}{4})_{\mathbf{h}(\frac{3\pi}{4})}$ (in blue), and $T_{\mathbf{h}}(\frac{3\pi}{4})$ (in purple) is given below:



(b) Differentiating $\mathbf{k}$ yields

$$\mathbf{k}'(t) = (\cos t - t \sin t,\ \sin t + t \cos t).$$

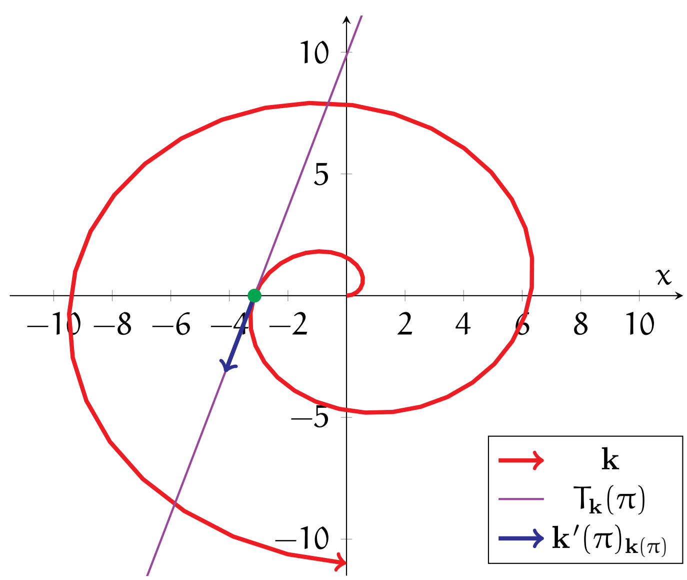
Evaluating $\mathbf{k}$ and $\mathbf{k}'$ at $t = \pi$, we obtain

$$\mathbf{k}(\pi) = (-\pi,\ 0), \qquad \mathbf{k}'(\pi) = (-1 - \pi \cdot 0,\ 0 + \pi \cdot \cos \pi) = (-1, -\pi).$$

As a result, the tangent line to $\mathbf{k}$ at $t = \pi$ is

$$T_{\mathbf{k}}(\pi) = \left\{ s \cdot (-1, -\pi)_{(-\pi,0)} \,\middle|\, s \in \mathbb{R} \right\}.$$

Finally, a sketch of $\mathbf{k}$ (in red), $\mathbf{k}'(\pi)_{\mathbf{k}(\pi)}$ (in blue), and $T_{\mathbf{k}}(\pi)$ (in purple) is given below:



(2) *(Warm-up)* Let $\mathcal{C}$ denote the unit circle about the origin:

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Compute, at each of the points $\mathbf{p} \in \mathcal{C}$ listed below, the tangent line to $\mathcal{C}$:

(a) $\mathbf{p} = (1, 0)$.

(b) $\mathbf{p} = (0, 1)$.

(c) $\mathbf{p} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Let γ be the usual polar parametrisation of $\mathcal{C}$:

$$\gamma : \mathbb{R} \to \mathcal{C}, \qquad \gamma(t) = (\cos t, \sin t).$$

Recall in particular that

$$\gamma'(t) = (-\sin t, \cos t), \qquad t \in \mathbb{R}.$$

We will use γ below to perform the necessary computations.

(a) First, notice that $\mathbf{p} = (1, 0)$ corresponds to $t = 0$:

$$\gamma(0) = (1, 0).$$

In addition, we have that

$$\gamma'(0) = (0, 1).$$

As a result, the tangent line at $\mathbf{p}$ is

$$T_{\mathbf{p}}\mathcal{C} = T_\gamma(0) = \{s \cdot \gamma'(0)_{\gamma(0)} \mid s \in \mathbb{R}\} = \{s \cdot (0, 1)_{(1,0)} \mid s \in \mathbb{R}\}.$$

(b) Notice that $\mathbf{p} = (0, 1)$ corresponds to $t = \frac{\pi}{2}$, i.e.

$$\gamma\left(\frac{\pi}{2}\right) = (0, 1), \qquad \gamma'\left(\frac{\pi}{2}\right) = (-1, 0).$$

As a result, the tangent line at $\mathbf{p}$ is

$$T_{\mathbf{p}}\mathcal{C} = T_\gamma\left(\frac{\pi}{2}\right) = \{s \cdot (-1, 0)_{(0,1)} \mid s \in \mathbb{R}\}.$$

(c) Notice that $\mathbf{p} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ corresponds to $t = \frac{5\pi}{6}$, i.e.

$$\gamma\left(\frac{5\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \qquad \gamma'\left(\frac{5\pi}{6}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

As a result, the tangent line at $\mathbf{p}$ is

$$T_p\mathcal{C} = T_\gamma\left(\frac{5\pi}{6}\right) = \left\{ s \cdot \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)_{\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)} \,\middle|\, s \in \mathbb{R} \right\}.$$

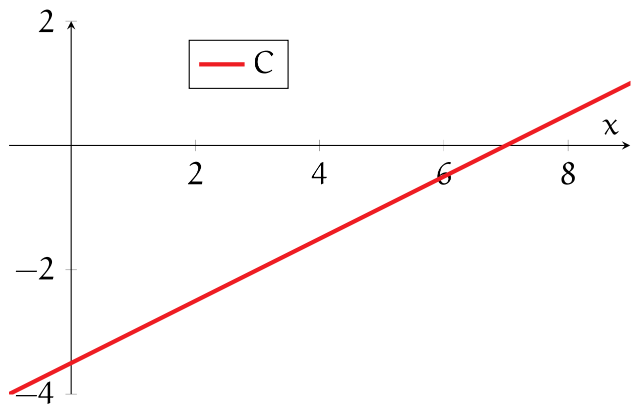
(3) *(Am I a curve?)* For each of the sets C provided below: (i) give a sketch of C, (ii) determine whether C is a curve or not, and (iii) justify your answer.

(a) $C = \{(x, y) \in \mathbb{R}^2 \mid x - 2y = 7\}$.

(b) $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 \mid x = y^2\}$.

(a) A sketch of C is given below:



Note that C is the level set $f(x, y) = 7$, where

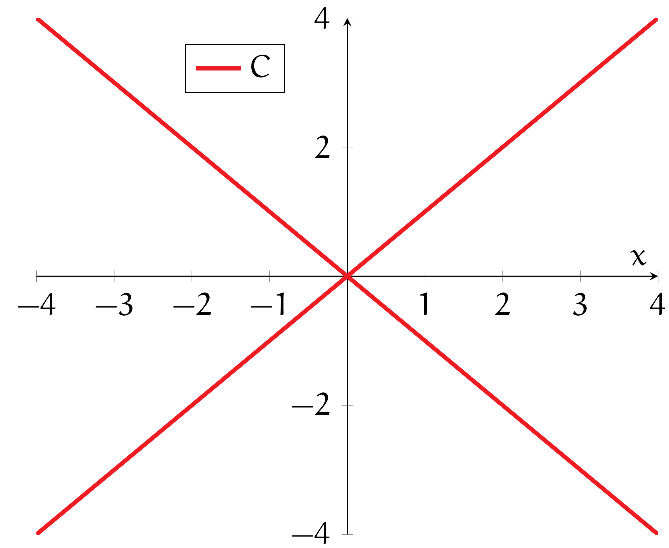
$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x - 2y.$$

Moreover, note that the gradient of f satisfies

$$\nabla f(x, y) = (1, -2)_{(x,y)},$$

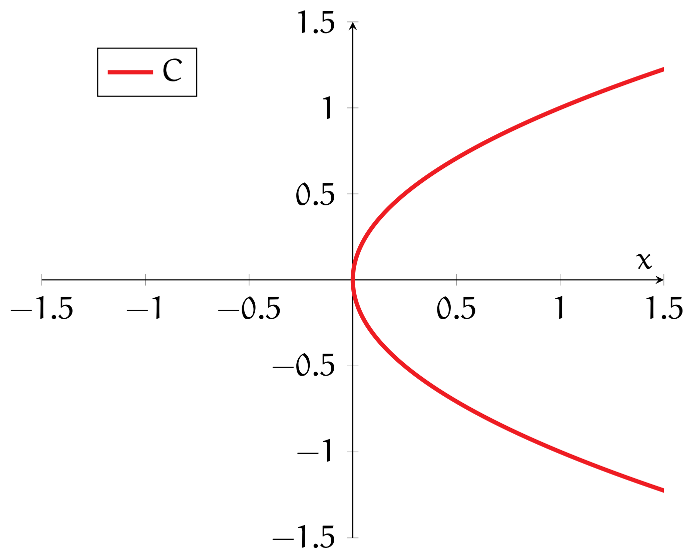
which is everywhere nonvanishing. As a result, C must be a curve.

(b) A sketch of C is given below:



Note that C is the union of two lines (namely, $x + y = 0$ and $x - y = 0$). In particular, C contains a "self-intersection" at the origin (where C looks like an "X", rather than a deformed interval). Thus, C is not a curve.

(c) A sketch of C is given below:



Observe that C is the level set $g(x, y) = 0$, where

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x - y^2.$$

Since the gradient of g satisfies

$$\nabla g(x, y) = (1, -2y)_{(x,y)},$$

which does not vanish at any point, then C must be a curve.

(4) [**Marked**] Consider the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - xy = 2\}.$$

(a) Show that C is a curve.

(b) Find a parametrisation of C whose image includes the point $(-1, 1)$. Be sure to specify the domain of your parametrisation.

(c) Find the tangent line to C at $(-1, 1)$.

(a) First, notice that C can be expressed as a level set

$$C = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = 2\}$$

where h is the function

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad h(x, y) = x^2 - xy.$$

Taking the gradient of h, we see that

$$\nabla h(x, y) = (2x - y, \, -x)_{(x,y)}.$$

[1 mark for getting (almost) this far]

By a bit of linear algebra, we see that the only solution to the linear system

$$2x - y = 0, \qquad -x = 0$$

is the trivial solution $(x, y) = (0, 0)$. Thus, it follows that $\nabla h(x, y) \neq (0, 0)_{(x,y)}$ except when $(x, y) = (0, 0)$. Notice, however, that $(0, 0) \notin C$, since

$$h(0, 0) = 0 \neq 2.$$

Thus, $\nabla h(\mathbf{p})$ is nonvanishing for every $\mathbf{p} \in C$, and it follows, from the level set theorem, that C is indeed a curve. [1 mark for getting this mostly correct]

(b) There are many ways to parametrise C. One straightforward way is to take "$x = t$". Then, by the definition of C, it follows that the y-component must satisfy

$$t^2 - ty = 2, \qquad y = t - \frac{2}{t},$$

the latter when $t \neq 0$. Using the above, we can define our parametrisation as

$$\gamma : (-\infty, 0) \to C, \qquad \gamma(t) = \left(t,\, t - \frac{2}{t}\right).$$

Note in particular that, as desired,

$$\gamma(-1) = (-1, 1).$$

(Observe that the interval $(-\infty, 0)$ is chosen so the formula for γ makes sense. Note γ still makes sense if the domain is $(0, \infty)$, but then γ would not pass through the point $(-1, 1)$.) [1 mark for a correct domain] [1 mark for correct parametric formula]

One could instead take "$y = t$", which results in the alternate parametrisation

$$\lambda : \mathbb{R} \to C, \qquad \lambda(t) = \left(\frac{t - \sqrt{t^2 + 8}}{2},\, t\right).$$

(c) We use the parametrisation γ from (b) here. First, we compute the following:

$$\gamma(-1) = (-1, 1), \qquad \gamma'(t) = \left(1,\, 1 + \frac{2}{t^2}\right), \qquad \gamma'(-1) = (1, 3). \quad \text{[1 mark]}$$

By definition, the tangent line to C through $(2, 2)$ is

$$T_{(-1,1)}C = T_\gamma(-1) = \left\{s \cdot \gamma'(-1)_{\gamma(-1)} \,\middle|\, s \in \mathbb{R}\right\} = \{s \cdot (1,3)_{(-1,1)} \mid s \in \mathbb{R}\}.$$

[1 mark for mostly correct answer]

(5) [Tutorial] Consider the *ellipse* given by

$$E = \{(x, y) \in \mathbb{R}^2 \mid 3x^2 + 2y^2 = 6\}.$$

(a) Show that E is a curve.

(b) Find a parametrisation of E that passes through the point $(-\sqrt{2}, 0) \in \mathsf{E}$.

(c) Find the tangent line to E at $(-\sqrt{2}, 0)$.

(a) First, observe that E can be expressed as a level set,

$$E = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 6\},$$

where f is the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = 3x^2 + 2y^2.$$

Now, the gradient of f satisfies

$$\nabla f(x, y) = (6x, 4y)_{(x,y)},$$

which vanishes only when $(x, y) = (0, 0)$. Since $(0, 0) \notin E$ (by definition), it follows that $\nabla f(x, y)$ is nonzero at all points of E. As a result, the level set theorem implies E is a curve.

(b) There are many ways to do this. The most direct method would be to set "$y = t$" and use the defining equation $3x^2 + 2y^2 = 6$ to determine x. Note that

$$y = t, \qquad x^2 = 2 - \frac{2}{3} \cdot t^2, \qquad x = \pm\sqrt{2 - \frac{2}{3} \cdot t^2}.$$

Since we need this parametrisation to pass through $(-\sqrt{2}, 0)$, we choose the negative branch of the above. Moreover, since the quantity under the square root is only defined when $|t| \leq \sqrt{3}$, we can then define our parametrisation by

$$\lambda : \left(-\sqrt{3}, \sqrt{3}\right) \to E, \qquad \lambda(t) = \left(-\sqrt{2 - \frac{2}{3} \cdot t^2},\ t\right).$$

In particular, λ does pass through $(-\sqrt{2}, 0) \in E$, as

$$\lambda(0) = \left(-\sqrt{2}, 0\right).$$

A slicker method (which only applies to ellipses) is to consider rescaled variables

$$\tilde{x} = \sqrt{3} \cdot x, \qquad \tilde{y} = \sqrt{2} \cdot y,$$

which then satisfy

$$\tilde{x}^2 + \tilde{y}^2 = 6.$$

In particular, the above describes a circle of radius $\sqrt{6}$, which can be parametrised as

$$\tilde{x} = \sqrt{6} \cdot \cos t, \qquad \tilde{y} = \sqrt{6} \cdot \sin t, \qquad t \in \mathbb{R}.$$

Changing back in terms of x and y then yields

$$x = \sqrt{2} \cdot \cos t, \qquad y = \sqrt{3} \cdot \sin t, \qquad t \in \mathbb{R}.$$

This gives us our desired parametrisation, which we can write as

$$\gamma : \mathbb{R} \to E, \qquad \gamma(t) = \left(\sqrt{2} \cdot \cos t,\ \sqrt{3} \cdot \sin t \right).$$

In particular, γ does pass through $(-\sqrt{2}, 0) \in E$, since $\gamma(\pi) = (-\sqrt{2}, 0)$.

(c) We can use either parametrisation from part (b) to find the tangent line, however the computations are far simpler using γ. First, we observe that

$$\gamma(\pi) = \left(-\sqrt{2}, 0 \right), \qquad \gamma'(t) = \left(-\sqrt{2} \cdot \sin t, \sqrt{3} \cdot \cos t \right), \qquad \gamma'(\pi) = \left(0, -\sqrt{3} \right).$$

As a result, we conclude that

$$T_{(-\sqrt{2},0)} E = T_{\gamma}(\pi) = \left\{ s \cdot \left(0, -\sqrt{3} \right)_{(-\sqrt{2},0)} \,\middle|\, s \in \mathbb{R} \right\}.$$

(6) [**Tutorial**] Mr Error recently attempted an *Introduction to Differential Geometry* problem sheet and did quite poorly. He definitely needs some help! Here, you can assume the role of a TA for *MTH5113* and help Mr Error see the error of his ways.

(a) Consider the following parabola in $\mathbb{R}^2$:

$$P = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}.$$

The following are two different (correct) parametrisations of P:

$$\gamma : \mathbb{R} \to P, \qquad \gamma(t) = (t, t^2),$$
$$\lambda : \mathbb{R} \to P, \qquad \lambda(u) = (u+1, (u+1)^2).$$

Mr Error computed the tangent line to γ at $t = 0$ and obtained

$$T_\gamma(0) = \left\{ s \cdot (1, 0)_{(0,0)} \,\middle|\, s \in \mathbb{R} \right\}.$$

He then computed the tangent line to λ at $u = 0$ and obtained

$$T_\lambda(0) = \left\{ s \cdot (1, 2)_{(1,1)} \,\middle|\, s \in \mathbb{R} \right\}.$$

Mr Error noticed that $T_\gamma(0)$ and $T_\lambda(0)$ were not the same, and concluded that tangent lines are not independent of parametrisation! Where did Mr Error err?

(b) Next, Mr Error considered the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^4 = 0\}.$$

He noticed that C is a level set of the function $f(x, y) = x^2 + y^4$ and hence deduced that C is curve. Moreover, he observed that C consists of only a single point,

$$C = \{(0, 0)\},$$

hence he concludes that a single point at the origin must be a curve (as we defined it in this module)! How did Mr Error go so far astray?

(a) The mistake is that Mr Error (correctly) computed the tangent line at two different points of the parabola. For γ, he computed the tangent line at $\gamma(0) = (0, 0)$, while for λ, he computed the tangent line at $\lambda(0) = (1, 1)$. Since these are two different points of the parabola, the two tangent lines will not be the same.

To demonstrate that the tangent line is independent of parametrisation, Mr Error would need to compute the tangent lines of γ and λ at the same point, e.g. at $\gamma(0) = (0, 0) = \lambda(-1)$.

(b) Mr Error's mistake is that he applied the theorem stating that level sets are curves without checking its main assumption: that the gradient ∇f is nonzero. In fact, since

$$\nabla f(x, y) = (2x, 4y^3)_{(x,y)},$$

then $\nabla f(0, 0)$ vanishes. As a result, the level sets theorem cannot be applied at the origin.

(7) *(Parametrise me!)* Consider the following curves:

$$\begin{aligned} C_1 &= \{(x, y) \in \mathbb{R}^2 \mid x = y^4\}, \\ C_2 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x = 0\}, \\ C_3 &= \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1,\ x + y - z = 2\}. \end{aligned}$$

(You can assume you already know each of the above is a curve.)

(a) Give one parametrisation of C_1 whose image is all of C_1.

(b) Give one parametrisation of C_2 whose image is all of C_2.

(c) Give one parametrisation of C_3 whose image is all of C_3.

(a) The easiest way to parametrise C_1 is to simply set the parameter t to be the y-variable (which varies over all of $\mathbb{R}$). Since $x = y^4$, the x-component must then be t^4:

$$\gamma_1 : \mathbb{R} \to C_1, \qquad \gamma_1(t) = (t^4, t).$$

(b) Here, the trick is to understand what the set C_2 is. The condition $x^2 + y^2 + z^2 = 1$ restricts you to the unit sphere—the set of all points of distance 1 from the origin. Adding the restriction $x = 0$ puts you also on the yz-plane.

Combining these two constraints, one then sees that C_2 is simply the unit circle about the origin on the yz-plane. As usual, the easiest way to parametrise circles is to use cosines and sines, now only in the y- and z-components:

$$\gamma_2 : \mathbb{R} \to C_2, \qquad \gamma_2(t) = (0, \cos t, \sin t).$$

(c) The first thing to notice is that each of the constraints in the definition of C_3,

$$x + y + z = 1, \qquad x + y - z = 2 \tag{1}$$

defines a plane in $\mathbb{R}^3$. Thus, the two equations together yield the intersection of these two planes, which is a line. The main task is to find what this line C_3 is, and to parametrise it.

The simplest way to find parametric equations is to combine the two equations in ways that simplify them. If we add the two equations in (1), we obtain

$$2(x+y) = 3, \qquad x + y = \frac{3}{2}. \tag{2}$$

If we subtract one equation in (1) from the other, we obtain

$$2z = -1, \qquad z = -\frac{1}{2}. \tag{3}$$

To parametrise C_3, we set the z-component to be the constant $-\frac{1}{2}$, and we relate the x- and y-components as in (2). In particular, if we set the x-component to be the parameter t, then the y-component must be $\frac{3}{2} - t$. Thus, one valid parametrisation of (all of) C_3 is

$$\gamma_3 : \mathbb{R} \to C_3, \qquad \gamma_3(t) = \left(t, \frac{3}{2} - t, -\frac{1}{2} \right).$$

A more systematic way to extract (2)–(3) from (1) is to apply the Gauss–Jordan elimination algorithm you learned from linear algebra:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right] \Rightarrow_{R2-R1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 \end{array}\right] \Rightarrow_{R1+\frac{1}{2}R2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & -2 & 1 \end{array}\right].$$

(8) *(Bad function? No problem!)* Find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

satisfies the following: (i) $\nabla f(x, y)$ vanishes for every $(x, y) \in C$, but (ii) C is a curve.

There are many possible functions f. One simple example is

$$f(x, y) = x^2.$$

For this particular f, we have that

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}.$$

Thus, C is simply the y-axis, which we know is a curve.

Moreover, the gradient of f satisfies

$$\nabla f(x, y) = (2x, 0)_{(x,y)}.$$

In particular, at every point of C (where $x = 0$), we have that ∇f vanishes.

(9) *(Level set theorem, in 3-d!)* Recall that one can often show that subsets of $\mathbb{R}^2$ are curves by showing that they are "good" level sets of functions. In fact, there is a corresponding result for subsets of $\mathbb{R}^3$, though the statement is a bit more complicated:

> **Theorem.** Let $U \subseteq \mathbb{R}^3$ be open and connected, and let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ both be smooth functions. Also, let $c_f, c_g \in \mathbb{R}$, and let C be the level set
>
> $$C = \{(x, y, z) \in U \mid f(x, y, z) = c_f,\ g(x, y, z) = c_g\}.$$
>
> If $\nabla f(\mathbf{p}) \times \nabla g(\mathbf{p})$ is nonzero for every $\mathbf{p} \in C$, then C is a curve.

Using this theorem, show that the following subsets of $\mathbb{R}^3$ are curves:

(a) $C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2,\ y = x\}$.

(b) $C_2 = \{(\cos t, \sin t, t) \mid t \in \mathbb{R}\}$.

(a) Observe that C_1 can be expressed as

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ g(x, y, z) = 0\},$$

where f and g are the functions

$$\begin{aligned} f &: \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = z - x^2 - y^2, \\ g &: \mathbb{R}^3 \to \mathbb{R}, \qquad g(x, y, z) = y - x. \end{aligned}$$

Next, we compute the gradients of f and g at each $(x, y, z) \in \mathbb{R}^3$:

$$\nabla f(x, y, z) = (-2x,\ -2y,\ 1)_{(x,y,z)}, \qquad \nabla g(x, y, z) = (-1,\ 1,\ 0)_{(x,y,z)}.$$

In particular, taking their cross product yields

$$\nabla f(x, y, z) \times \nabla g(x, y, z) = (-1,\ -1,\ -2x - 2y)_{(x,y,z)} \neq \mathbf{0}_{(x,y,z)},$$

for any $(x, y, z) \in \mathbb{R}^3$. Thus, by the theorem, C_1 is indeed a curve.

(b) The main point is to notice that C_2 can be written as

$$\begin{aligned} C_2 &= \{(x, y, z) \in \mathbb{R}^3 \mid x = \cos z,\ y = \sin z\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ g(x, y, z) = 0\}, \end{aligned}$$

where f and g are given by

$$\begin{aligned} f &: \mathbb{R}^3 \to \mathbb{R}, & f(x, y, z) &= x - \cos z, \\ g &: \mathbb{R}^3 \to \mathbb{R}, & g(x, y, z) &= y - \sin z. \end{aligned}$$

Now, for any $(x, y, z) \in \mathbb{R}^3$, the gradients of f and g satisfy

$$\nabla f(x, y, z) = (1,\ 0,\ \sin z)_{(x,y,z)}, \qquad \nabla g(x, y, z) = (0,\ 1,\ -\cos z)_{(x,y,z)}.$$

and their cross product is

$$\nabla f(x, y, z) \times \nabla g(x, y, z) = (-\sin z,\ \cos z,\ 1)_{(x,y,z)} \neq \mathbf{0}_{(x,y,z)}.$$

Thus, the theorem implies that C_2 is a curve.