# **MTH5113** (Winter 2022): Problem Sheet 2
*Solutions*

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**(1)** *(Warm-up)* Compute each of the following:

**(a)** Consider the vector-valued function

$$\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^2, \qquad \mathbf{f}(t) = (t^2, t^3 - 1).$$

(i) Compute $\mathbf{f}'(t)$ for every $t \in \mathbb{R}$.

(ii) Find the values $\mathbf{f}'(0)$, $\mathbf{f}'(1)$, and $\mathbf{f}'(-2)$.

**(b)** Consider the vector-valued function

$$\mathbf{g} : (0, 1) \rightarrow \mathbb{R}^3, \qquad \mathbf{g}(t) = (\ln t, \ln(1 - t), e^{3t} + t).$$

(i) What happens to $\mathbf{g}(t)$ as $t$ approaches $0$? As $t$ approaches $1$?

(ii) Compute $\mathbf{g}'(t)$ for every $t \in \mathbb{R}$.

(iii) Compute the second derivative $\mathbf{g}''(t)$ for every $t \in \mathbb{R}$.

**(a)** These are direct computations:

(i) To find $\mathbf{f}'(t)$, we differentiate each component of $\mathbf{f}$:

$$\mathbf{f}'(t) = \left( \frac{d}{dt}(t^2), \frac{d}{dt}(t^3 - 1) \right) = (2t, 3t^2).$$

(ii) We substitute $t = 0$, $t = 1$, and $t = -2$ into the preceding formula:

$$\begin{aligned} \mathbf{f}'(0) &= (2 \cdot 0, 3 \cdot 0^2) = (0, 0), \\ \mathbf{f}'(1) &= (2 \cdot 1, 3 \cdot 1^2) = (2, 3), \\ \mathbf{f}'(-2) &= (2 \cdot (-2), 3 \cdot (-2)^2) = (-4, 12). \end{aligned}$$

**(b)** These are also direct computations:

(i) As $t$ approaches $0$, the $x$-component ($\ln t$) of $\mathbf{g}(t)$ tends to $-\infty$, while the $y$- and $z$-components have finite limits ($0$ and $1$, respectively):

$$\lim_{t \searrow 0} \mathbf{g}(t) = (-\infty, 0, 1).$$

Also, as $t$ approaches $1$, the $y$-component ($\ln(1-t)$) of $\mathbf{g}(t)$ tends to $-\infty$, while the $x$- and $z$-components have finite limits ($0$ and $e^3 + 1$, respectively).

$$\lim_{t \nearrow 1} \mathbf{g}(t) = (0, -\infty, e^3 + 1).$$

(ii) Recalling the calculus identities

$$\frac{d}{dt}(\ln t) = \frac{1}{t}, \qquad \frac{d}{dt}(e^t) = e^t, \qquad t \in (0, 1),$$

as well as the chain rule, we obtain that

$$\begin{aligned} \mathbf{g}'(t) &= \left( \frac{d}{dt}(\ln t),\ \frac{d}{dt}[\ln(1-t)],\ \frac{d}{dt}(e^{3t} + t) \right) \\ &= \left( \frac{1}{t},\ \frac{1}{1-t} \cdot \frac{d}{dt}(1-t),\ e^{3t} \cdot \frac{d}{dt}(3t) + 1 \right) \\ &= \left( \frac{1}{t},\ \frac{1}{t-1},\ 3e^{3t} + 1 \right). \end{aligned}$$

(iii) To compute $\mathbf{g}''(t)$, we differentiate the preceding formula yet again:

$$\begin{aligned} \mathbf{g}''(t) &= \left( \frac{d}{dt}\left(\frac{1}{t}\right),\ \frac{d}{dt}\left(\frac{1}{t-1}\right),\ \frac{d}{dt}(3e^{3t} + 1) \right) \\ &= \left( -\frac{1}{t^2},\ -\frac{1}{(t-1)^2},\ 9e^{3t} \right). \end{aligned}$$

**(2)** *(Warm-up)* Let $\mathbf{A}$ denote the vector-valued function

$$\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \mathbf{A}(x, y, z) = (x(1-z), y(1-z)).$$

**(a)** Compute the partial derivatives $\partial_1 \mathbf{A}(x, y, z)$, $\partial_2 \mathbf{A}(x, y, z)$, and $\partial_3 \mathbf{A}(x, y, z)$ at every point $(x, y, z) \in \mathbb{R}^3$.

**(b)** Find $\partial_2 \mathbf{A}(1, 0, 3)$ and $\partial_3 \mathbf{A}(0, -1, -1)$.

**(a)** To find $\partial_1 \mathbf{A}(x, y, z)$, we take the corresponding derivative of each component of $\mathbf{A}$:

$$\partial_1 \mathbf{A}(x, y, z) = (\partial_x[x(1-z)], \partial_x[y(1-z)]) = (1-z, 0).$$

In the last step, we treated $y$ and $z$ as constants and differentiated with respect to $x$. The remaining partial derivatives of $\mathbf{A}$ are computed analogously:

$$\partial_2 \mathbf{A}(x, y, z) = (\partial_y[x(1-z), \partial_y[y(1-z)]]) = (0, 1-z),$$
$$\partial_3 \mathbf{A}(x, y, z) = (\partial_z[x(1-z), \partial_z[y(1-z)]]) = (-x, -y).$$

*(Make sure your answers are 2-dimensional vectors!)*

**(b)** Substituting $(x, y, z) = (1, 0, 3)$ to the above formula for $\partial_2 \mathbf{A}(x, y, z)$ yields

$$\partial_2 \mathbf{A}(1, 0, 3) = (0, 1-3) = (0, -2).$$

Similarly, substituting $(x, y, z) = (0, -1, -1)$ into the formula for $\partial_3 \mathbf{A}(x, y, z)$ yields

$$\partial_3 \mathbf{A}(0, -1, -1) = (-0, -(-1)) = (0, 1).$$

**(3)** *(Warm-up)* Let $\mathbf{F}$ be the vector field on $\mathbb{R}^2$ defined via the formula

$$\mathbf{F}(x, y) = (x - y, x + y)_{(x,y)}.$$

**(a)** Compute the following: (i) $\mathbf{F}(1, -1)$; (ii) $\mathbf{F}(-2, -1)$; (iii) $\mathbf{F}(-1, \frac{1}{2})$.

**(b)** Plot the three tangent vectors from part (a) onto a Cartesian plane.

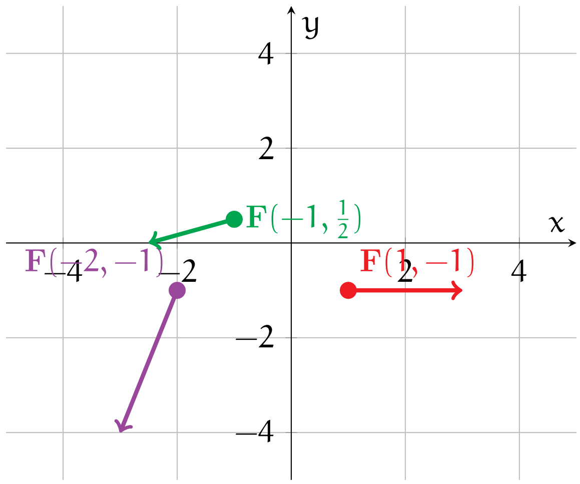
**(a)** Each of these is a direct computation:

(i) $\mathbf{F}(1, -1) = (1 - (-1), 1 + (-1))_{(1,-1)} = (2, 0)_{(1,-1)}$.

(ii) $\mathbf{F}(-2, -1) = (-2 - (-1), -2 + (-1))_{(-2,-1)} = (-1, -3)_{(-2,-1)}$.

(iii) $\mathbf{F}(-1, \frac{1}{2}) = (-1 - \frac{1}{2}, -1 + \frac{1}{2})_{(-1,\frac{1}{2})} = (-\frac{3}{2}, -\frac{1}{2})_{(-1,\frac{1}{2})}$.

**(b)** The tangent vectors are drawn below:




**(4)** [**Tutorial**] Consider the following vector-valued function:

$$\mathbf{h} : (0, \infty) \to \mathbb{R}^2, \qquad \mathbf{h}(t) = (t \cos t,\ t \sin t).$$

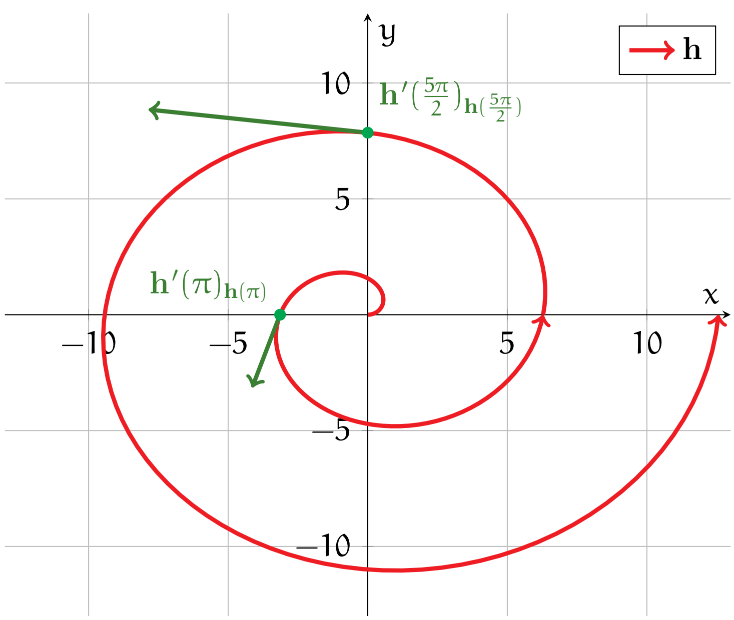
**(a)** Sketch the values $\mathbf{h}(t)$, for all $0 < t < 4\pi$. Also, plot the values of $\mathbf{h}$ on computer (see the *Additional Resources* section on the *QMPlus* page).

**(b)** Compute $\mathbf{h}(\pi)$ and $\mathbf{h}(\frac{5\pi}{2})$.

**(c)** Compute $\mathbf{h}'(\pi)$ and $\mathbf{h}'(\frac{5\pi}{2})$.

**(d)** Draw $\mathbf{h}'(\pi)_{\mathbf{h}(\pi)}$ and $\mathbf{h}'\left(\frac{5\pi}{2}\right)_{\mathbf{h}(\frac{5\pi}{2})}$ on your sketch in part (a).

**(a)** The sketch is below, with the image of $\mathbf{h}$ drawn in red.



**(b)** The desired values of $\mathbf{h}$ are below:

$$\mathbf{h}(\pi) = (\pi \cos \pi, \; \pi \sin \pi) = (-\pi, 0),$$
$$\mathbf{h}\left(\frac{5\pi}{2}\right) = \left(\frac{5\pi}{2} \cos \frac{5\pi}{2}, \frac{5\pi}{2} \sin \frac{5\pi}{2}\right) = \left(0, \frac{5\pi}{2}\right),$$

**(c)** Taking a derivative of $\mathbf{h}$ (using the product rule) yields

$$\mathbf{h}'(t) = (\cos t - t \sin t, \sin t + t \cos t).$$

In particular, setting $t = \pi$ and $t = \frac{5\pi}{2}$ yields

$$\mathbf{h}'(\pi) = (\cos \pi - \pi \sin \pi, \; \sin \pi + \pi \cos \pi) = (-1, -\pi),$$
$$\mathbf{h}'\left(\frac{5\pi}{2}\right) = \left(\cos \frac{5\pi}{2} - \frac{5\pi}{2} \sin \frac{5\pi}{2}, \sin \frac{5\pi}{2} + \frac{5\pi}{2} \cos \frac{5\pi}{2}\right) = \left(-\frac{5\pi}{2}, 1\right),$$

**(d)** The tangent vectors

$$\mathbf{h}'(\pi)_{\mathbf{h}(\pi)} = (-1, \pi)_{(-\pi, 0)}, \qquad \mathbf{h}'\left(\frac{5\pi}{2}\right)_{\mathbf{h}\left(\frac{5\pi}{2}\right)} = \left(-\frac{5\pi}{2}, 1\right)_{\left(0, \frac{5\pi}{2}\right)},$$

are drawn as green arrows on the diagram in part (a).

**(5)** **[Marked]** Let $\beta$ be the vector-valued function

$$\beta : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \beta(u, v) = (2u^2 + 2v^2 - 1, v, u).$$

**(a)** Sketch the values $\beta(u, v)$, for all $0 < u < 1$ and $0 < v < 1$.

**(b)** On the sketch in part (a), indicate (i) the path obtained by holding $u = \frac{1}{2}$ and varying $v$, and (ii) the path obtained by holding $v = \frac{1}{2}$ and varying $u$.

**(c)** Compute the following quantities:

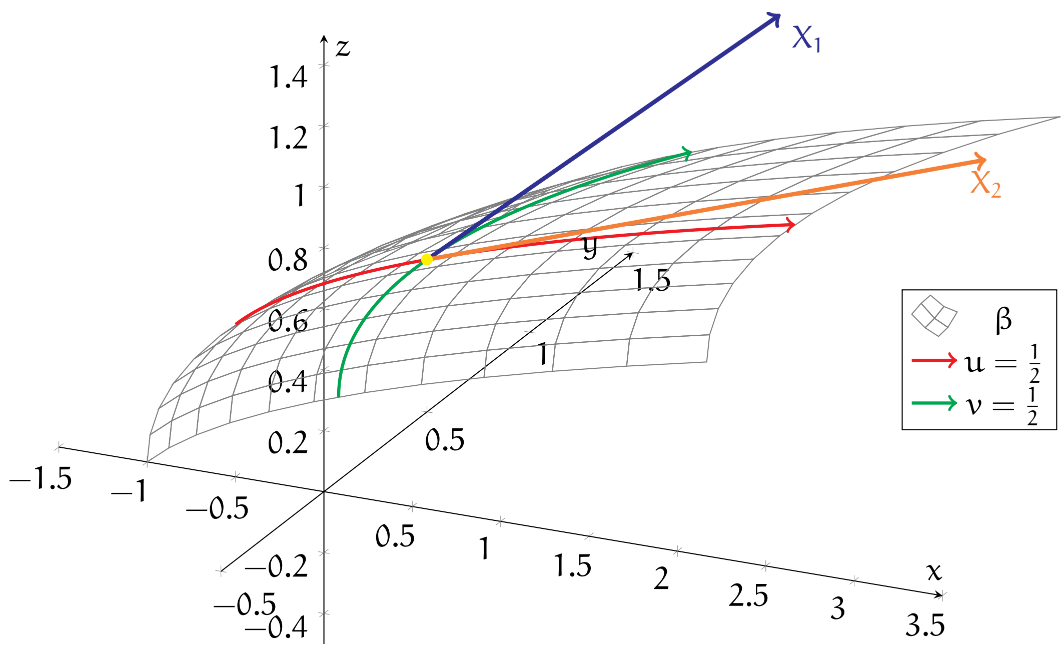
$$\beta\left(\frac{1}{2}, \frac{1}{2}\right), \qquad \partial_1 \beta\left(\frac{1}{2}, \frac{1}{2}\right), \qquad \partial_2 \beta\left(\frac{1}{2}, \frac{1}{2}\right).$$

**(d)** Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\beta\left(\frac{1}{2},\frac{1}{2}\right)_{\beta\left(\frac{1}{2},\frac{1}{2}\right)}, \qquad X_2 = \partial_2\beta\left(\frac{1}{2},\frac{1}{2}\right)_{\beta\left(\frac{1}{2},\frac{1}{2}\right)}.$$

**(a)** The image of $\beta$ is drawn in grey below. [1 mark for mostly correct drawing]

**(b)** The path in (i) is drawn below in red *(a parabolic segment)*, while the path in (ii) is drawn in green *(also a parabolic segment)*. [1 mark for mostly correct drawings]




**(c)** We first compute the partial derivatives of $\alpha$: [1 mark]

$$\partial_1\beta(u,v) = (4u, 0, 1), \qquad \partial_2\beta(u,v) = (4v, 1, 0).$$

Evaluating at $(u,v) = (\frac{1}{2},\frac{1}{2})$, we obtain [1 mark]

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \left(0,\frac{1}{2},\frac{1}{2}\right), \qquad \partial_1\beta\left(\frac{1}{2},\frac{1}{2}\right) = (2,0,1), \qquad \partial_2\beta\left(\frac{1}{2},\frac{1}{2}\right) = (2,1,0).$$

**(d)** These tangent vectors are drawn in the diagram from parts (a) and (b) ($X_1$ in blue, and $X_2$ in orange). [1 mark for mostly correct arrows]

**(6)** *(Compute 'n' plot)* Let $\lambda$ denote the vector-valued function

$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (t, t^2 - 1).$$

**(a)** Compute the following: $\lambda(-2)$, $\lambda(-1)$, $\lambda(0)$, $\lambda(1)$, and $\lambda(2)$.

**(b)** Compute the following: $\lambda'(-2)$, $\lambda'(-1)$, $\lambda'(0)$, $\lambda'(1)$, and $\lambda'(2)$.

**(c)** Sketch the values $\lambda(t)$, for all $-3 < t < 3$, on a Cartesian plane.

**(d)** Draw the following tangent vectors as arrows on your sketch in part (a):

$$\lambda'(-2)_{\lambda(-2)}, \qquad \lambda'(-1)_{\lambda(-1)}, \qquad \lambda'(0)_{\lambda(0)}, \qquad \lambda'(1)_{\lambda(1)}, \qquad \lambda'(2)_{\lambda(2)}.$$

**(a)** The desired values of $\lambda$ are below:

$$\begin{aligned}
\lambda(-2) &= (-2, (-2)^2 - 1) = (-2, 3), \\
\lambda(-1) &= (-1, (-1)^2 - 1) = (-1, 0), \\
\lambda(0) &= (0, 0^2 - 1) = (0, -1), \\
\lambda(1) &= (1, 1^2 - 1) = (1, 0), \\
\lambda(2) &= (2, 2^2 - 1) = (2, 3).
\end{aligned}$$

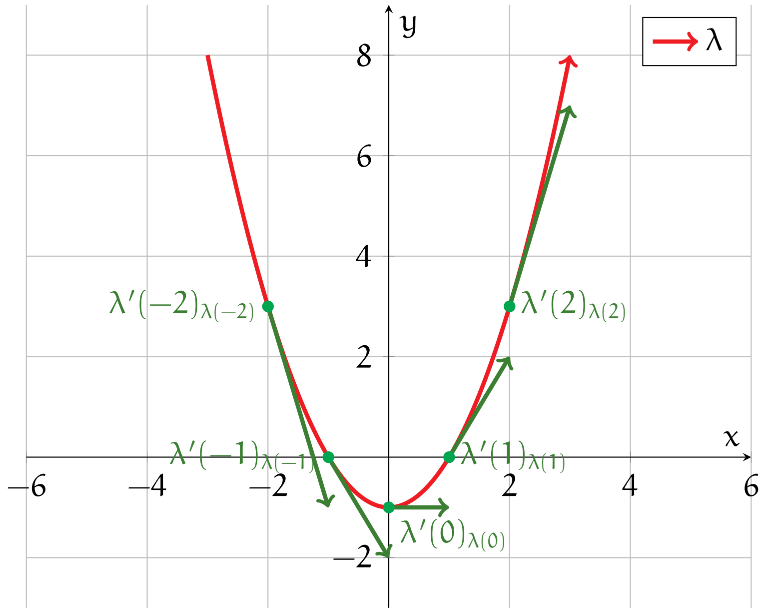
**(b)** First, note that the derivative of $\lambda$ satisfies

$$\lambda'(t) = (1, 2t).$$

Thus, taking $t = -2, -1, 0, 1, 2$, we obtain

$$\begin{aligned}
\lambda'(-2) &= (1, 2 \cdot (-2)) = (1, -4), \\
\lambda'(-1) &= (1, 2 \cdot (-1)) = (1, -2), \\
\lambda'(0) &= (1, 2 \cdot 0) = (1, 0), \\
\lambda'(1) &= (1, 2 \cdot 1) = (1, 2), \\
\lambda'(2) &= (1, 2 \cdot 2) = (1, 4).
\end{aligned}$$

**(c)** The sketch is below, with the image of $\lambda$ drawn in red. *(The most direct way to sketch* *this is to note that $\lambda$ is the graph of the parabolic function $f(x) = x^2 - 1$. In addition, you could use the answers in parts (a) and (b) to help you draw $\lambda$.)*




**(d)** From parts (b) and (c), we have that

$$\begin{aligned} \lambda'(-2)_{\lambda(-2)} &= (1, -4)_{(-2,3)}, \\ \lambda'(-1)_{\lambda(-1)} &= (1, -2)_{(-1,0)}, \\ \lambda'(0)_{\lambda(0)} &= (1, 0)_{(0,-1)}, \\ \lambda'(1)_{\lambda(1)} &= (1, 2)_{(1,0)}, \\ \lambda'(2)_{\lambda(2)} &= (1, 4)_{(2,3)}. \end{aligned}$$

These are drawn as green arrows on the diagram in part (c).

**(7)** *(Compute 'n' plot II)* Consider the vector-valued function

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

*(See also Question 8 from Problem Sheet 1.)*

**(a)** Sketch the image of $\sigma$. (Use a computer to help if needed; see the *Additional Resources* section on the *QMPlus* page)

**(b)** On the sketch in part (a), indicate (i) the path obtained by holding $u = \frac{\pi}{2}$ and varying $v$, and (ii) the path obtained by holding $v = \frac{\pi}{2}$ and varying $u$.

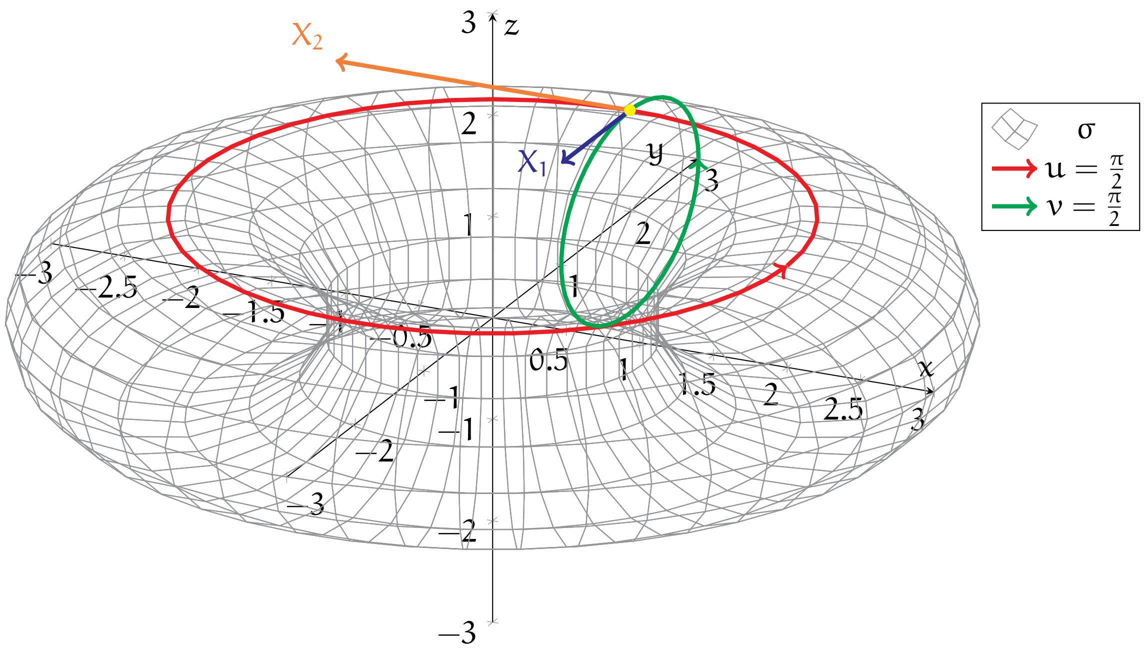
**(c)** Compute the partial derivatives $\partial_1\sigma(u,v)$ and $\partial_2\sigma(u,v)$ for all $(u,v) \in \mathbb{R}^2$.

**(d)** Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2},\frac{\pi}{2})}, \qquad X_2 = \partial_2\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2},\frac{\pi}{2})}.$$

**(a)** A sketch is given below part (b), with the image of $\sigma$ drawn in grey.

**(b)** The path in (i) is drawn below in red, while the path in (ii) is drawn in green.




**(c)** To find $\partial_1\sigma(u,v)$ and $\partial_2\sigma(u,v)$, we differentiate each component:

$$\partial_1\sigma(u,v) = (-\sin u \cos v, -\sin u \sin v, \cos u),$$
$$\partial_2\sigma(u,v) = (-(2+\cos u)\sin v, (2+\cos u)\cos v, 0).$$

**(d)** First, we compute

$$\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0,2,1), \qquad \partial_1\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0,-1,0), \qquad \partial_2\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (-2,0,0).$$

As a result, we have that

$$X_1 = (0, -1, 0)_{(0,2,1)}, \qquad X_2 = (-2, 0, 0)_{(0,2,1)},$$

The tangent vectors $X_1$ and $X_2$ are drawn in the diagram from parts (a) and (b).

**(8)** *(Gradients 'n' plot)* Consider the function

$$p : \mathbb{R}^2 \to \mathbb{R}, \qquad p(x, y) = x - y^2.$$

**(a)** Sketch the following sets on a Cartesian plane:

(i) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\}$.

(ii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 2\}$.

(iii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = -2\}$.

**(b)** Compute the gradient $\nabla p(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

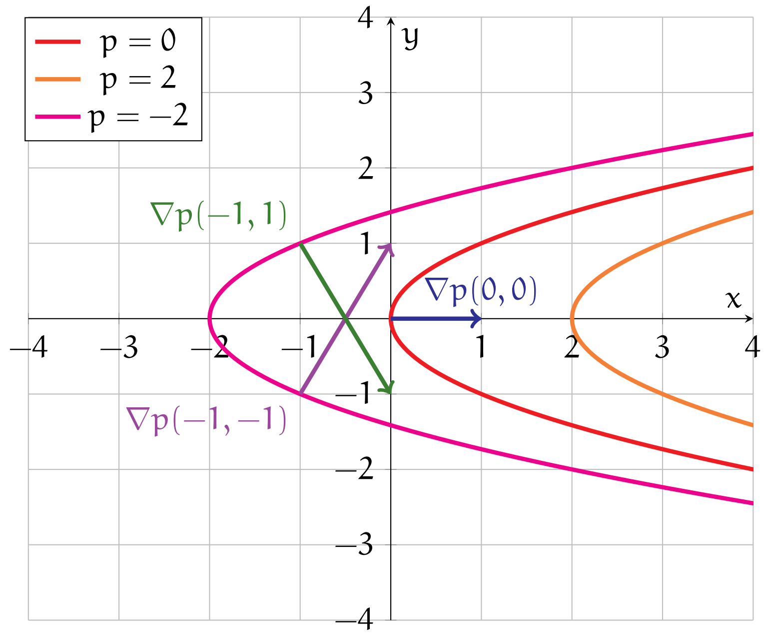
**(c)** Plot the following values onto your sketch from part (a):

(i) $\nabla p(0, 0)$.

(ii) $\nabla p(-1, -1)$.

(ii) $\nabla p(-1, 1)$.

**(a)** The three sets are sketched below in (i) red, (ii) orange, and (iii) pink:




**(b)** The partial derivatives of $p$ are

$$\partial_1 p(x, y) = 1, \qquad \partial_2 p(x, y) = -2y.$$

Thus, the gradient of $p$ is

$$\nabla p(x, y) = (\partial_1 p(x, y), \partial_2 p(x, y))_{(x,y)} = (1, -2y)_{(x,y)}.$$

**(c)** Substituting the appropriate values for $x$ and $y$, we obtain that

(i) $\nabla p(0, 0) = (1, 0)_{(0,0)}$.

(ii) $\nabla p(-1, -1) = (1, 2)_{(-1,-1)}$.

(iii) $\nabla p(-1, 1) = (1, -2)_{(-1,1)}$.

The corresponding arrows are drawn in the plot from (a) in (i) blue, (ii) purple, (iii) green.

**(9)** *(Connections to "Convergence and Continuity")* Consider the following subsets of $\mathbb{R}^2$:

$$V = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}, \qquad L = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}.$$

**(a)** Give an informal justification of the following: (i) $V$ is open; (ii) $L$ is not open.

**(b)** *(Not examinable)* Give a rigorous proof of the two statements in part (a).

**(c)** Is the following subset of $\mathbb{R}^2$ connected:

$$Q = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}?$$

Give a brief (informal) justification of your answer.

**(a)** Informal justifications for both statements are given below:

(i) Consider a point $(x, y) \in V$, so that $x > 0$. Suppose you take a step away from $(x, y)$, in any direction, to another point $(x', y')$. Then, as long as that step is small enough, we would still have $x' > 0$, and hence $(x', y') \in V$. Thus, by definition, $V$ is open.

(ii) Consider the point $(0, 0) \in L$. Suppose you take a step away from $(0, 0)$ in the $x$-direction. Then, no matter how small of a step you take, you will always no longer be on $L$. Thus, $L$ violates the definition of openness and hence is not open.

**(b)** Formal proofs of both statements are given below:

(i) To prove that $V$ is open, we must establish the following statement:

($*$) *For any* $(x, y) \in V$, *there exists* $\delta > 0$ *such that for any* $(x', y') \in \mathbb{R}^2$ *satisfying* $|(x', y') - (x, y)| < \delta$, *we have* $(x', y') \in V$.

Let $(x, y)$ be an arbitrary element of $V$; note that $x > 0$. Moreover, let us choose $\delta = x > 0$. Then, given any $(x', y') \in \mathbb{R}^2$ such that $|(x', y') - (x, y)| < x$, we have that

$$x > |(x', y') - (x, y)| \geq |x' - x|,$$

and it follows that $x' > 0$. As a result, $(x', y') \in V$, and hence ($*$) is proved.

(ii) Negating the definition of open subsets, we see that we must prove the following:

($\star$) *There exists some* $(x, y) \in L$ *such that for every* $\delta > 0$, *there exists* $(x', y') \in \mathbb{R}^2$ *such that* $|(x', y') - (x, y)| < \delta$, *but* $(x', y') \notin L$.

Let us choose $(x, y) = (0, 0) \in L$. Given an arbitrary $\delta > 0$, we choose the point $(x', y') = (\frac{\delta}{2}, 0)$. In particular, we have that $(\frac{\delta}{2}, 0) \notin L$, and that

$$|(x', y') - (x, y)| = \left| \left( \frac{\delta}{2}, 0 \right) - (0, 0) \right| = \frac{\delta}{2} < \delta.$$

In particular, the above proves the statement $(\star)$.

**(c)** The set $Q$ is not connected.

To justify this, we consider two points $(x_1, y_1), (x_2, y_2) \in Q$, with $y_1 < 0 < y_2$. Then, any path that connects $(x_1, y_1)$ to $(x_2, y_2)$ must pass through the (horizontal) line $y = 0$ (this comes from the intermediate value theorem), and hence this path must leave $Q$.

**(10)** *(Good derivative, bad derivative)*

- **(a)** *(Not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for some $(x, y)$.
- **(b)** *(Fun! But not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for *any* $(x, y)$.

**(a)** One example of such a function $b$ is the following:

$$b : \mathbb{R}^2 \to \mathbb{R}, \qquad b(x, y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

Note that $b$ is always constant if we hold $y$ constant and vary only with respect to $x$. As a result, $\partial_1 b(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

On the other hand, if we fix $x = 0$, for instance, and we vary in $y$, we see that

$$b(0, y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

In particular, this fails to be continuous at $y = 0$, hence we cannot differentiate with respect to $y$ there. As a result, $\partial_2 b(0, 0)$ fails to exist.

**(b)** One example of such a function $b$ is the following:

$$b : \mathbb{R}^2 \to \mathbb{R}, \qquad b(x, y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 0 & \text{if } y \notin \mathbb{Q}. \end{cases}$$

(Here, $\mathbb{Q}$ is the set of rational numbers.)

Again, $b$ is always constant if we hold $y$ constant and vary only with respect to $x$. As a result, $\partial_1 b(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$. On the other hand, if we fix any $x$-value and vary in $y$, then the resulting function $y \mapsto b(x, y)$ fails to be continuous at any value of $y$. As a result, $\partial_2 b(x, y)$ cannot exist at any $(x, y) \in \mathbb{R}^2$.